

On the Cauchy problem for differential operators with double characteristics, a transition from effective to non-effective characteristics

Tatsuo Nishitani

Abstract

We discuss the well-posedness of the Cauchy problem for hyperbolic operators with double characteristics which changes from non-effectively hyperbolic to effectively hyperbolic, on the double characteristic manifold, across a submanifold of codimension 1. We assume that there is no bicharacteristic tangent to the double characteristic manifold and the spatial dimension is 2. Then we prove the well-posedness of the Cauchy problem in all Gevrey classes assuming, on the double characteristic manifold, that the ratio of the imaginary part of the subprincipal symbol to the real eigenvalue of the Hamilton map is bounded and that the sum of the real part of the subprincipal symbol and the modulus of the imaginary eigenvalue of the Hamilton map is strictly positive.

1 Introduction

This paper is a continuation of our previous papers [15, 16]. Let

$$P(x, D) = -D_0^2 + A_1(x, D')D_0 + A_2(x, D')$$

be a differential operator of order 2 in D_0 with coefficients $A_j(x, D')$, classical pseudodifferential operator of order j on \mathbb{R}^n depending smoothly on x_0 where $x = (x_0, x') = (x_0, x_1, \dots, x_n)$. We assume that the principal symbol $p(x, \xi)$ of $P(x, D)$ vanishes exactly of order 2 on a C^∞ manifold Σ and

$$(1.1) \quad \text{rank} \left(\sum_{j=0}^n d\xi_j \wedge dx_j \Big|_{\Sigma} \right) = \text{constant}.$$

As in [15, 16] we assume that $\text{codim } \Sigma = 3$ and

$$(1.2) \quad \begin{cases} \text{the spectral structure of } F_p \text{ changes simply} \\ \text{across a submanifold } S \text{ of codimension 1 of } \Sigma. \end{cases}$$

By conjugation with a Fourier integral operator one can assume $A_1 = 0$ then, near any point $\rho \in \Sigma$, one can write

$$p(x, \xi) = -\xi_0^2 + \phi_1(x, \xi')^2 + \phi_2(x, \xi')^2$$

where $d\phi_1$ and $d\phi_2$ are linearly independent at ρ and $\Sigma = \{\xi_0 = 0, \phi_1 = 0, \phi_2 = 0\}$. Under the assumptions (1.1) and (1.2) without restrictions we can assume (see [15])

$$\{\xi_0, \phi_2\} > 0, \quad \{\xi_0, \phi_1\} = O(|\phi|)$$

near ρ . Here and in what follows $f = O(|\phi|)$, $\phi = (\phi_1, \phi_2)$ means that f is a linear combination of ϕ_1 and ϕ_2 near the reference point. We first recall

Lemma 1.1 ([16, Lemma 1.2]) *If the spectral structure of F_p changes across S then we have $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = 0$ on S and one of the following cases occurs;*

- (i) $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 < 0$ in $\Sigma \setminus S$ so that p is non-effectively hyperbolic in Σ with $\text{Ker}F_p^2 \cap \text{Im}F_p^2 = \{0\}$ in $\Sigma \setminus S$ and $\text{Ker}F_p^2 \cap \text{Im}F_p^2 \neq \{0\}$ on S ,
- (ii) $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 > 0$ in $\Sigma \setminus S$ so that p is effectively hyperbolic in $\Sigma \setminus S$ and non-effectively hyperbolic on S with $\text{Ker}F_p^2 \cap \text{Im}F_p^2 \neq \{0\}$,
- (iii) $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2$ changes the sign across S , that is p is effectively hyperbolic in the one side of $\Sigma \setminus S$, non-effectively hyperbolic in the other side with $\text{Ker}F_p^2 \cap \text{Im}F_p^2 = \{0\}$ and non-effectively hyperbolic on S with $\text{Ker}F_p^2 \cap \text{Im}F_p^2 \neq \{0\}$.

Let us denote

$$\Sigma^\pm = \{(x, \xi) \in \Sigma \mid \pm(\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2) > 0\}.$$

Since the eigenvalues of F_p are 0 and $\pm\sqrt{\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2}$ on Σ so that F_p has non-zero real eigenvalues on Σ^+ and non-zero pure imaginary eigenvalues on Σ^- in the case (iii). Let us set

$$2\kappa(\rho)^2 = |\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2|$$

and we make precise the meaning “simply” in (1.2), namely we assume that there is $C > 0$ such that

$$(1.3) \quad \text{dist}_\Sigma(\rho, S)/C \leq \kappa(\rho) \leq C \text{dist}_\Sigma(\rho, S)$$

on Σ where $\text{dist}_\Sigma(\rho, S)$ denotes the distance from ρ to S on Σ . Our aim in this paper is to complete the proof of the following result:

Theorem 1.2 *Assume (1.2) and that there is no bicharacteristic tangent to Σ and there exist $\epsilon > 0$, $C > 0$ such that*

$$(1.4) \quad (1 - \epsilon)\mu(\rho) + \text{Re}P_{\text{sub}}(\rho) \geq \epsilon, \quad |\text{Im}P_{\text{sub}}(\rho)| \leq Ce(\rho), \quad \rho \in \Sigma \cap \{|\xi| = 1\}$$

where $\pm e(\rho)$ ($e(\rho) \geq 0$) are real eigenvalues and $\pm i\mu(\rho)$ ($\mu(\rho) \geq 0$) are pure imaginary eigenvalues of $F_p(\rho)$. We also assume $n = 2$ in the case (iii). Then the Cauchy problem for P is well-posed in any Gevrey class $\gamma^{(s)}$ for $s > 1$.

The case (i) in Theorem 1.2, namely $e(\rho) \equiv 0$ on Σ was proved in [4] while in [15], it was proved under less restrictive assumption, the non existence of bicharacteristics tangent to S . The case (ii) in Theorem 1.2 and hence $\mu(\rho) \equiv 0$ on Σ , was proved in [16]. Some transition cases from effectively hyperbolic to non-effectively hyperbolic are studied in [3, 1, 5]. In particular in [1, 5] a typical case of (iii) was studied but the condition (1.4) was not investigated. In this paper we give a proof of Theorem 1.2 for the case (iii) assuming $n = 2$, while if $n = 1$ the case $\text{Ker} F_p^2 \cap \text{Im} F_p^2 \neq \{0\}$ never occur.

Remark 1.3 For differential operators, the condition (1.4) with $\epsilon = 0$ can be expressed as

$$\text{dist}_{\mathbb{C}}(P_{\text{sub}}(\rho), [-\text{Tr}^+ F_p(\rho), \text{Tr}^+ F_p(\rho)]) \leq C e(\rho)$$

which generalizes the Ivrii-Petkov-Hörmander condition ([8, 6]) and R.Melrose conjectured in [12] that this condition is necessary for the Cauchy problem to be C^∞ well-posed, but little is known about necessary conditions for the well-posedness when the spectral structure of F_p changes.

Remark 1.4 With $X^\pm = \{\xi_0, \phi_2\} H_{\xi_0} - \{\phi_1, \phi_2\} H_{\phi_1} \pm \sqrt{2} \kappa(\rho) H_{\phi_2}$ it is easy to see

$$F_p(\rho) X^\pm = \pm e(\rho) X^\pm, \quad \rho \in \Sigma^+$$

and there exist exactly two bicharacteristics passing ρ transversally to Σ^+ with tangents X^\pm (see [11]). Since $d\phi_2(X^\pm) = \{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = 2\kappa(\rho)^2 > 0$ this implies that the surface $\phi_2 = 0$ is spacelike on Σ^+ . On the other hand there is no bicharacteristic reaching Σ^- (see [9]).

2 Idea of the proof of Theorem 1.2

From Lemma 1.1 we have $\text{Ker} F_p^2 \cap \text{Im} F_p^2 \neq \{0\}$ on Σ^- and there is no bicharacteristic tangent to Σ^- by assumption. Then thanks to [14, Theorem 3.3] p admits an elementary decomposition microlocally at every point on Σ^- . As in [4, 15] we try to decompose $p = -(\xi_0 + \phi_1 - \psi)(\xi_0 - \phi_1 + \psi) + q$ with $\psi = o(|\phi_1|)$ and non-negative q verifying $\{\xi_0 - \phi_1 + \psi, q\} = O(q)$ in Σ^- . These requirements essentially determine ψ and actually the non existence of tangent bicharacteristic assures that $\xi_0 - \psi_1 + \psi$ commutes against q better than the usual case. On the other hand, as checked in Remark 1.4 the surface $\hat{\phi}_2 = 0$ is spacelike on Σ^+ , then [13, 16] suggests the use of pseudodifferential weight $T \approx e^{\zeta \log \hat{\phi}_2}$ where ζ is a cutoff symbol to Σ^+ . Our strategy for proving Theorem 1.2 is rather naive so that we make a such decomposition and derive weighted energy estimates with the cutoff weight T . But the decomposition should be compatible with the cutoff weights and to achieve this goal we must be careful in choosing cutoff symbols and in estimating errors caused by them. The assumption $n = 2$ enables us to choose all symbols which we need, including cutoff symbols, in $S_{3/4, 1/2}$ and we carry out pseudodifferential calculus within the framework of $S_{3/4, 1/2}$ though we often need the calculus in smaller class than $S_{3/4, 1/2}$.

In the rest of this section we express the assumptions in more explicit form. In what follows we assume $n = 2$ and we work in a conic neighborhood of $\bar{\rho} \in S$. Without restrictions we may assume $\bar{\rho} = (0, \mathbf{e}_3)$, $\mathbf{e}_3 = (0, 0, 1) \in \mathbb{R}^3$ with a system of local coordinates $x = (x_0, x') = (x_0, x_1, x_2)$. From (1.3) and Lemma 1.1 one can write

$$(2.1) \quad \{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = \theta|\xi'| + c_1\phi_1 + c_2\phi_2$$

in a neighborhood of $\bar{\rho}$ where S is defined by $\{\theta = 0\} \cap \Sigma$ and $d\theta \neq 0$ on S and hence $\Sigma^\pm = \Sigma \cap \{\pm\theta > 0\}$. Compare this to the case (i) and (ii) where we have $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = \mp\theta^2 + c_1\phi_1 + c_2\phi_2$ respectively ([15, 16]). Here note that

$$e(\rho) = \begin{cases} \sqrt{2}\kappa(\rho) & \rho \in \Sigma^+ \\ 0 & \rho \in \Sigma^- \end{cases}, \quad \mu(\rho) = \begin{cases} 0 & \rho \in \Sigma^+ \\ \sqrt{2}\kappa(\rho) & \rho \in \Sigma^- \end{cases}.$$

Since $\{\xi_0, \phi_2\}^2 - \{\phi_1, \phi_2\}^2 = \{\xi_0 - \phi_1, \phi_2\}\{\xi_0 + \phi_1, \phi_2\} = 0$ on S we may assume without restrictions that

$$(2.2) \quad \{\xi_0 - \phi_1, \phi_2\} = 0 \quad \text{on } S$$

and $\{\xi_0, \phi_2\} = \{\phi_1, \phi_2\} > 0$ on S (see [15, 16]).

Lemma 2.1 *In a conic neighborhood of $\bar{\rho}' = (0, \mathbf{e}_2)$ one can assume that*

$$\phi_2(x, \xi') = \hat{\phi}_2(x)e(x, \xi'), \quad \theta(x, \xi')|\xi_2|^{-1} = \psi(x') + f(x, \xi')\hat{\phi}_2(x)$$

where $0 \neq e(x, \xi') \in S_{1,0}^1$ and $f(x, \xi') \in S_{1,0}^0$. Moreover we have $\{\theta, \phi_j\} = c_j\phi_2$ with $c_j \in S_{1,0}^0$.

Proof: Since $\{\xi_0, \phi_2\} \neq 0$ then one can write $\phi_2 = (x_0 - \psi_2(x, \xi'))b_2$ where ψ_2 is independent of x_0 and $b_2 \neq 0$. From $\{\phi_1, \phi_2\} \neq 0$ we see $\{\psi_2, \phi_1\} \neq 0$. This shows that $d\psi_2$ is not proportional to $\sum_{j=0}^2 \xi_j dx_j$ at $\bar{\rho}$ because otherwise we would have $\phi_1(0, \mathbf{e}_2) = \partial\phi_1(0, \mathbf{e}_2)/\partial\xi_2 \neq 0$. Since $\Xi_0 = \xi_0$, $X_0 = x_0$, $X_1 = \psi_2$ verifies the commutation relations and $d\Xi_0, dX_0, dX_1, \sum_{j=0}^2 \xi_j dx_j$ are linearly independent at $\bar{\rho}$, just observed above, these coordinates extends to homogeneous symplectic coordinates (X, Ξ) (see [6, Theorem 21.1.9]). Switching the notation to (x, ξ) we can assume that $\phi_2 = (x_0 - x_1)e$. Since $\{\phi_2, \phi_1\} \neq 0$ one can write $\phi_1 = (\xi_1 - \psi_1)b_1$ where ψ_1 is independent of ξ_0 and ξ_1 . Writing $\psi_1(x, \xi_2) = \bar{\psi}_1(x', \xi_2) + e_1\phi_2$ and $\theta(x, \xi') = \tilde{\theta}(x', \xi_2) + (x_0 - x_1)\theta_1 + (\xi_1 - \bar{\psi}_1)\theta_2$ so that S is given by $\xi_0 = 0, x_0 - x_1 = 0, \xi_1 - \bar{\psi}_1(x', \xi_2) = 0, \tilde{\theta}(x', \xi_2) = 0$ where $\tilde{\theta}(x', \xi_2) = \theta(x_1, x_1, x_2, \bar{\psi}_1(x', \xi_2), \xi_2)$. Since $\tilde{\theta}$ is of homogeneous of degree 1 in ξ_2 one can write

$$\tilde{\theta}(x', \xi_2) = \tilde{\theta}(x', 1)\xi_2 = \psi(x')\xi_2$$

in a conic neighborhood of $(0, \mathbf{e}_2)$, where we have used the assumption $n = 2$. Let us set $\theta = \psi(x')\xi_2 + (\{\psi(x')\xi_2, \phi_1\}/\{\phi_1, \phi_2\})\phi_2$ then it is clear that $\{\theta, \phi_j\} = c_j\phi_2$ and hence this θ is a desired one. \square

Remark 2.2 Since the restriction $n = 2$ is only used to prove Lemma 2.1 then Theorem 1.2 is still true if we can choose a homogeneous symplectic coordinates such that Lemma 2.1 holds.

We now assume that ϕ_2 and θ satisfy Lemma 2.1 and set

$$\hat{\theta} = \theta|\xi_2|^{-1}, \quad \hat{\phi}_1 = \phi_1|\xi'|^{-1}$$

so that $\hat{\theta}$ and $\hat{\phi}_1$ are homogeneous of degree 0 in ξ' . From (2.2) we can write

$$(2.3) \quad \{\xi_0 - \phi_1, \hat{\phi}_2\} = \hat{c}\hat{\theta} + c'_1\hat{\phi}_1 + c'_2\hat{\phi}_2$$

near $\bar{\rho}$ where $\hat{c} > 0$ which follows from (2.1). Since we have $\{\xi_0 + \phi_1, \phi_2\}|\hat{c}\hat{\theta}||e| = 2\kappa^2$ on Σ and $\{\xi_0 + \phi_1, \phi_2\}/2\{\phi_1, \phi_2\} = 1$ on S then for any $\epsilon > 0$ there is a neighborhood of $\bar{\rho}$ where we have

$$(2.4) \quad (1 - \epsilon)\kappa^2(\rho) \leq \{\phi_1, \phi_2\}|\hat{c}\hat{\theta}||e| \leq (1 + \epsilon)\kappa^2(\rho).$$

Here we examine how the non existence of tangent bicharacteristics reflects on the Poisson brackets of symbols .

Proposition 2.1 ([16, Proposition 2.1]) *Assume $\{\theta, \phi_j\} = O(|\phi|)$ and that there is no bicharacteristic tangent to Σ . Then we have*

$$\{\xi_0, \theta\}(\rho) = 0, \quad \{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\}(\rho) = 0, \quad \rho \in S.$$

Lemma 2.3 *Assume that $\{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\} = 0$ on S . Then one can write $\{\xi_0 - \phi_1, \hat{\phi}_2\} = \hat{c}\hat{\theta} + c_0\hat{\theta}\hat{\phi}_1 + c_1\hat{\phi}_1^2 + c_2\hat{\phi}_2$.*

Lemma 2.4 *Assume that $\{\xi_0, \hat{\theta}\} = 0$, $\{\{\xi_0 - \phi_1, \phi_2\}, \phi_2\} = 0$ on S . Then we have $\{\xi_0 - \phi_1, \hat{\theta}\} = c_0\hat{\theta} + c_1\hat{\phi}_1^2 + c_2\hat{\phi}_2$.*

Proof: Note that $\{\xi_0 - \phi_1, \hat{\theta}\} = \alpha\hat{\theta} + \beta\hat{\phi}_1 + \gamma\hat{\phi}_2$. On the other hand we see

$$\{\hat{\theta}, \{\xi_0 - \phi_1, \phi_2\}\} = O(|\hat{\phi}|), \quad \{\xi_0 - \phi_1, \{\hat{\theta}, \phi_2\}\} = O(|(\hat{\theta}, \hat{\phi})|).$$

Then from the Jacobi identity it follows that $\beta = O(|(\hat{\theta}, \hat{\phi})|)$ and hence we have $\{\xi_0 - \phi_1, \hat{\theta}\} = \alpha\hat{\theta} + c_0\hat{\theta}\hat{\phi}_1 + c_1\hat{\phi}_1^2 + c_2\hat{\phi}_2$ which proves the assertion. \square

Corollary 2.5 *We have $\{\xi_0, \hat{\theta}\} = c_0\hat{\theta} + c_1\hat{\phi}_1^2 + c_2\hat{\phi}_2$.*

3 Cutoff and weight symbols

We use the same notation as in [16]. We first make a dilation of the coordinate x_0 ; $x_0 \rightarrow \mu x_0$ with small $\mu > 0$ so that $P(x, \xi, \mu) = \mu^2 P(\mu x_0, x', \mu^{-1}\xi_0, \xi')$ will be

$$\begin{aligned} p(\mu x_0, x', \xi_0, \mu \xi') + \mu P_1(\mu x_0, x', \xi_0, \xi') + \mu^2 P_0(\mu x_0, x') \\ = p(x, \xi, \mu) + P_1(x, \xi, \mu) + P_0(x, \mu). \end{aligned}$$

In what follows we often express such symbols dropping μ . It is easy to see that $a(\mu x_0, x', \mu \xi') = a(x, \xi', \mu) \in S(\langle \mu \xi' \rangle^m, g_0)$ if $a(x, \xi') \in S_{1,0}^m$ where $g_0 = |dx|^2 + \langle \xi' \rangle_\mu^{-2} |d\xi'|^2$. To prove the well-posedness of the Cauchy problem, applying [17, Theorem 1.1], it suffices to derive energy estimates for $P_{\xi'}$ which coincides with original P in a conic neighborhood of $(0, 0, \xi')$, $|\xi'| = 1$. Thus we can assume that the following conditions are satisfied globally;

$$(3.1) \quad \begin{cases} p(x, \xi) = -\xi_0^2 + \phi_1(x, \xi')^2 + \phi_2(x, \xi')^2, & \phi_j \in S(\langle \mu \xi' \rangle, g_0), \\ \{\xi_0, \phi_1\} = d_1 \phi_1 + d_2 \phi_2, & d_j \in \mu S(1, g_0), \\ \{\xi_0 - \phi_1, \hat{\phi}_2\} = \mu \hat{c} \hat{\theta} + c_0 \hat{\theta} \hat{\phi}_1 + c_1 \hat{\phi}_1^2 + c_2 \hat{\phi}_2, \\ \{\xi_0, \hat{\theta}\} = c'_0 \hat{\theta} + c'_1 \hat{\phi}_1^2 + c'_2 \hat{\phi}_2, \\ \{\phi_1, \hat{\phi}_2\} \geq c\mu, & c > 0 \end{cases}$$

where $c_j, c'_j \in \mu S(1, g_0)$ and $\hat{\theta} \in S(1, g_0)$ verifies

$$(3.2) \quad \{\hat{\theta}, \phi_j\} = c_j \hat{\phi}_2, \quad c_j \in \mu S(1, g_0)$$

and $\sup |\hat{\theta}|, \sup |\hat{\phi}_j|$ can be assumed to be sufficiently small, shrinking a conic neighborhood of $(0, 0, \xi')$ where we are working.

Let us put $P_{sub} = P_1^s + iP_2^s$ with real $P_i^s \in \mu S(\langle \mu \xi' \rangle, g_0)$ then from (1.4) and (2.4) the following conditions can be assumed to be satisfied globally;

$$(3.3) \quad \begin{cases} \mu^{1/2} \sqrt{\hat{c} \{\phi_1, \hat{\phi}_2\} |\hat{\theta}| |e|} + P_1^s \geq c\mu \langle \mu \xi' \rangle, & \hat{\theta} < 0, \quad P_1^s \geq c\mu \langle \mu \xi' \rangle, \quad \hat{\theta} > 0, \\ P_2^s = \mu c_0 \hat{\theta} \langle \mu \xi' \rangle + c_{11} \phi_1 + c_{12} \phi_2 \end{cases}$$

with a constant $c > 0$ and $c_0 \in S(1, g_0)$, $c_{ij} \in \mu S(1, g_0)$ where $c_0 = 0$ for $\hat{\theta} < 0$. Recall from [16]

$$\begin{cases} \phi = \langle \xi' \rangle_\mu^{1/2} (\hat{\phi}_2 + w), \\ \Phi = \pi + i \{ \log(\hat{\phi}_2 + i\omega) - \log(\hat{\phi}_2 - i\omega) \} = \pi - 2 \arg(\hat{\phi}_2 + i\omega), \\ w = (\hat{\phi}_2^2 + \langle \xi' \rangle_\mu^{-1})^{1/2}, \quad \omega = (\hat{\phi}_1^4 + \langle \xi' \rangle_\mu^{-1})^{1/2}, \\ \rho^2 = \hat{\phi}_2^2 + \omega^2 = \hat{\phi}_2^2 + \hat{\phi}_1^4 + \langle \xi' \rangle_\mu^{-1} \geq (w^2 + \omega^2)/2 \end{cases}$$

where ϕ plays a major role in our arguments and Φ is introduced in order to manage the energy estimates in the region $C\hat{\phi}_1^2 \geq w$. Note that

$$(3.4) \quad \{F, \Phi\} = 2(\omega \{F, \hat{\phi}_2\} - \hat{\phi}_2 \{F, \omega\}) / \rho^2.$$

We use the following metrics

$$\begin{cases} g = w^{-2} |dx|^2 + w^{-1} \langle \xi' \rangle_\mu^{-2} |d\xi'|^2, \\ g_1 = (\rho^{-1} + \omega^{-1/2})^2 |dx|^2 + \omega^{-1} \langle \xi' \rangle_\mu^{-2} |d\xi'|^2, \\ \tilde{g} = (w^{-1} + \omega^{-1/2})^2 |dx|^2 + \langle \xi' \rangle_\mu^{-3/2} |d\xi'|^2, \\ \bar{g} = \langle \xi' \rangle_\mu^{-1} |dx|^2 + \langle \xi' \rangle_\mu^{-3/2} |d\xi'|^2. \end{cases}$$

Note that $g, g_1 \leq \tilde{g} \leq \bar{g}$ and \bar{g} is the metric defining the class $S_{3/4,1/2}$ for any fixed $\mu > 0$. As checked in [16], we have $\omega \in S(\omega, g_1)$, $\rho \in S(\rho, g_1)$ and $\Phi \in S(1, g_1)$. With cutoff symbol $\zeta(x, \xi') = \zeta(\hat{\theta}w^{-1})$ we define the following weight

$$(3.5) \quad T = \exp(n\zeta^2(\chi^2 \log \phi + \Phi))$$

where $\chi = \chi(\hat{\phi}_1^2 w^{-1})$ and $\zeta(s) = 1$ in $s \geq -b_1$ and $\zeta(s) = 0$ in $s \leq -b_2$ with $\zeta'(s) \geq 0$ and n is a positive parameter.

$$\begin{array}{ccccccc} 1 & \chi & \chi_2 & \zeta_- & \zeta & & 1 \\ & & & & & \zeta_+ & \\ 0 & d_1 & d_2 & d_3 & -b_3 & -b_2 & -b_1 & 0 & b_1 & b_2 & b_3 \end{array}$$

Let $\zeta_{\pm}(x, \xi') = \zeta_{\pm}(\hat{\theta}w^{-1})$ and $\chi_2(x, \xi') = \chi_2(\hat{\phi}_1^2 w^{-1})$ where $\zeta_{\pm}(s) = 1$ in $\pm s \geq b_3$ and 0 in $\pm s \leq b_2$ so that $\zeta\zeta_+ = \zeta_+$ and $\zeta\zeta_- = 0$. We simply write χ, χ_2 for $\chi(x, \xi')$ and $\chi_2(x, \xi')$ and ζ, ζ_{\pm} for $\zeta(x, \xi')$ and $\zeta_{\pm}(x, \xi')$ if there is no confusions. It is easy to check $\chi, \chi_2 \in S(1, g)$. As for new cutoff symbols ζ, ζ_{\pm} we have

Lemma 3.1 *Let $G = w^{-2}|dx|^2 + \langle \xi' \rangle_{\mu}^{-2}|d\xi'|^2$ ($\leq g$) then $w \in S(w, G)$ and $\phi \in S(\phi, G)$. We have also $\zeta, \zeta_{\pm} \in S(1, G)$. Let $s \in \mathbb{R}$ then $\zeta_+ \hat{\theta}^s \in S(|\hat{\theta}|^s, G)$. Moreover if $0 < s \leq 1$ and $|\alpha| \neq 0$ we have $|(\zeta_+ \hat{\theta}^s)_{\beta}^{(\alpha)}| \leq C_{\alpha\beta} w^s \langle \xi' \rangle_{\mu}^{-|\alpha|} w^{-|\beta|}$.*

Proof: To prove $\phi \in S(\phi, G)$, with $\tilde{\phi} = \hat{\phi}_2 + w$, it is enough to show $\tilde{\phi} \in S(\tilde{\phi}, G)$. Note that one can write

$$\partial_x^{\beta} \partial_{\xi'}^{\alpha} \tilde{\phi} = \frac{\partial_x^{\beta} \partial_{\xi'}^{\alpha} \hat{\phi}_2(x)}{w} \tilde{\phi} + \frac{\partial_x^{\beta} \partial_{\xi'}^{\alpha} \langle \xi' \rangle_{\mu}^{-1}}{2w} = b_{\alpha\beta} \tilde{\phi} + a_{\alpha\beta}$$

with $b_{\alpha\beta} \in S(w^{-|\beta|} \langle \xi' \rangle_{\mu}^{-|\alpha|}, G)$ and $a_{\alpha\beta} \in S((w^{-1} \langle \xi' \rangle_{\mu}^{-1}) w^{-|\beta|} \langle \xi' \rangle_{\mu}^{-|\alpha|}, G)$ for $|\alpha + \beta| = 1$. By induction on $|\alpha + \beta|$ we see easily $\partial_x^{\beta} \partial_{\xi'}^{\alpha} \tilde{\phi} = b_{\alpha\beta} \tilde{\phi} + a_{\alpha\beta}$ with $b_{\alpha\beta} \in S(w^{-|\beta|} \langle \xi' \rangle_{\mu}^{-|\alpha|}, G)$ and $a_{\alpha\beta} \in S((w^{-1} \langle \xi' \rangle_{\mu}^{-1}) w^{-|\beta|} \langle \xi' \rangle_{\mu}^{-|\alpha|}, G)$ for any α, β . Since $w^{-1} \langle \xi' \rangle_{\mu}^{-1} \leq 2\tilde{\phi}$ we get the assertion. To prove $\zeta \in S(1, G)$ it suffices to show that

$$(3.6) \quad |\zeta' \partial_x^{\beta} \partial_{\xi'}^{\alpha} (\hat{\theta}w^{-1})| \leq C_{\alpha\beta} w^{-|\beta|} \langle \xi' \rangle_{\mu}^{-|\alpha|}.$$

By Lemma 2.1 without restrictions we may assume $\hat{\theta}(x, \xi') = \psi(x') + f(x, \xi') \hat{\phi}_2(x)$ from which it follows $|\partial_{\xi'}^{\alpha} \hat{\theta}| \leq C_{\alpha} \langle \xi' \rangle_{\mu}^{-|\alpha|} w$ for $|\alpha| \geq 1$. Noting $|\zeta' \hat{\theta}w^{-1}| \leq C$ we get (3.6). On the support of ζ_+ the estimate

$$|(\hat{\theta}^s)_{(\beta)}^{(\alpha)}| \leq \sum C_{\alpha_1, \dots, \beta_k} \hat{\theta}^s |\hat{\theta}_{(\beta_1)}^{(\alpha_1)}| \hat{\theta}^{-1} \dots |\hat{\theta}_{(\beta_k)}^{(\alpha_k)}| \hat{\theta}^{-1}$$

holds where $|\alpha_i + \beta_i| \geq 1$ and $\alpha_1 \dots + \alpha_k = \alpha, \beta_1 \dots + \beta_k = \beta$. On the other hand Lemma 2.1 shows that $|\hat{\theta}_{(\beta_i)}^{(\alpha_i)}| \leq C_{\alpha_i \beta_i} \langle \xi' \rangle_{\mu}^{-|\alpha_i|} w^{1-|\beta_i|}$ if $|\alpha_i| \neq 0$ and bounded by

C_{β_i} if $|\alpha_i| = 0$. Since $\hat{\theta}^{-1}w$ is bounded on the support of ζ_+ the third assertion is clear. If $|\alpha_i| \neq 0$ then noting $\hat{\theta}^s |\hat{\theta}_{(\beta_i)}^{(\alpha_i)} \hat{\theta}^{-1}| \leq C_{\alpha_i \beta_i} w^s \langle \xi' \rangle_\mu^{-|\alpha_i|} w^{-|\beta_i|}$ on the support of ζ_+ one gets the last assertion. \square

Remark 3.2 If $n > 2$ the $\psi(x')$ in Lemma 2.1 would depend on ξ' also and hence ζ, ζ_\pm does not belong to $S(1, g)$ in general.

To decompose p let us define

$$(3.7) \quad \psi = (-h\zeta_-^2 + \nu\zeta_+^2)\hat{\theta}\phi_1 + \chi_2\phi_1^3\langle\mu\xi'\rangle^{-2} = \tilde{\zeta}\hat{\theta}\phi_1 + \chi_2\phi_1^3\langle\mu\xi'\rangle^{-2}$$

with a positive parameter $0 < \nu \ll 1$ which will be determined later where $\tilde{\zeta} = -h\zeta_-^2 + \nu\zeta_+^2$ with $h = \mu\hat{c}\{\phi_1, \hat{\phi}_2\}^{-1} > 0$. Using ψ we rewrite p as

$$(3.8) \quad \begin{aligned} p &= -(\xi_0 + \phi_1 - \psi)(\xi_0 - \phi_1 + \psi) + 2\psi\phi_1 - \psi^2 + \phi_2^2 \\ &= -(\xi_0 + \phi_1 - \psi)(\xi_0 - \phi_1 + \psi) + q \end{aligned}$$

where

$$\begin{cases} q = \phi_2^2 + 2a^2\tilde{\zeta}\hat{\theta}\phi_1^2 + 2a^2\chi_2\phi_1^4\langle\mu\xi'\rangle^{-2}, \\ a = (1 - \tilde{\zeta}\hat{\theta}/2 - \chi_2\phi_1^2\langle\mu\xi'\rangle^{-2}/2)^{1/2}. \end{cases}$$

The main part of $\{\xi_0 - \phi_1 + \psi, q\}$ will be $\{\xi_0 - \phi_1 + \psi, \phi_2^2\}$ which is required to be $O(q)$ in $\theta < 0$ as explained above. Indeed, by our choice, we have

$$(3.9) \quad \begin{aligned} \{\xi_0 - \phi_1 + \psi, \hat{\phi}_2\} &= \mu(1 - \zeta_-^2)\hat{c}\hat{\theta} + \mu\nu\hat{h}^{-1}\hat{c}\zeta_+^2\hat{\theta} \\ &\quad + c_1\hat{\phi}_1^2 + c_2\hat{\theta}\hat{\phi}_1 + c_3\hat{\phi}_2 \end{aligned}$$

where $1 - \zeta_-^2 = 0$ in $\hat{\theta} \leq -b_3w$ so that $|(1 - \zeta_-^2)\hat{\theta}| \leq Cw$ in $\hat{\theta} \leq 0$.

Lemma 3.3 *We have $(\tilde{\zeta}\hat{\theta})_{(\beta)}^{(\alpha)}, (\chi_2\hat{\phi}_1^2)_{(\beta)}^{(\alpha)} \in S(\langle\xi'\rangle_\mu^{-|\alpha|}, g)$ for $|\alpha + \beta| = 1$. Hence the same holds for $a_{(\beta)}^{(\alpha)}$. In particular $|(\tilde{\zeta}\hat{\theta})_{(\beta)}^{(\alpha)}|, |(\chi_2\hat{\phi}_1^2)_{(\beta)}^{(\alpha)}|$ and $|a_{(\beta)}^{(\alpha)}|$ are bounded by $C_{\alpha\beta}w^{1/2}\langle\xi'\rangle_\mu^{-|\alpha|}w^{-|\alpha|/2-|\beta|}$ for $|\alpha + \beta| \geq 1$ and bounded by $C_{\alpha\beta}w\langle\xi'\rangle_\mu^{-|\alpha|}w^{-|\alpha|/2-|\beta|}$ for $|\alpha + \beta| \geq 2$.*

In this paper $Op(\phi)$ denotes the Weyl quantized pseudodifferential operator with symbol ϕ and we denote $Op(\phi)Op(\psi) = Op(\phi\#\psi)$. We often use the same letter to denote a symbol and the operator with such symbol if there is no confusion. Thus we denote

$$Op(\phi\psi)u = \phi\psi u, \quad Op(\phi)Op(\psi)u = \phi(\psi u).$$

We make some additional preparations (see [10]). Let $c = id_1 + ic_{11}$ with c_{11}, d_1 in (3.3), (3.2) and we set $M = \xi_0 + \phi_1 - \psi + c, \Lambda = \xi_0 - \phi_1 + \psi - c$ and write

$$p + P_1^s + iP_2^s = -M\#\Lambda + Q = -M\#\Lambda + q + T_1 + iT_2.$$

Note that $-(\xi_0 + \phi_1 - \psi)(\xi_0 - \phi_1 + \psi) = -M\Lambda - c\phi_1 - 2c\psi - c^2$. In view of Lemma 3.3 it is not difficult to check

$$M\#\Lambda = M\Lambda + i\{\xi_0, \phi_1 - \psi + c_{11}\} + c_1 w^{1/2} \phi_1 + c_2 \hat{\phi}_1^2 \langle \mu \xi' \rangle + R$$

with $c_i \in \mu S(1, \bar{g})$, $R \in \mu^2 S(w^{-1}, \bar{g})$. Therefore we see from (3.1) that T_1 satisfies

$$(3.10) \quad \begin{cases} \mu^{1/2} \sqrt{\hat{c}\{\phi_1, \hat{\phi}_2\}|\hat{\theta}|} |e(x, \xi')| + T_1 \geq 2\bar{\kappa}\mu \langle \mu \xi' \rangle, & \hat{\theta} < 0, \\ T_1 \geq 2\bar{\kappa}\mu \langle \mu \xi' \rangle, & \hat{\theta} > 0 \end{cases}$$

with some $\bar{\kappa} > 0$ and T_2 can be written

$$(3.11) \quad T_2 = \mu c_0 \hat{\theta} \langle \mu \xi' \rangle + b_0 \hat{\theta} \phi_1 + b_1 \hat{\phi}_1^2 \langle \mu \xi' \rangle + b_2 \phi_2 + b_3 w^{1/2} \phi_1$$

with $b_i \in S(1, \bar{g})$. Thus $T_2 \langle \mu \xi' \rangle^{-1} = O(|(\hat{\theta}, \hat{\phi}_1^2, \hat{\phi}_2, w^{1/2} \hat{\phi}_1)|)$, a linear combination without $\hat{\phi}_1$. We transform P by T so that

$$PT = T\tilde{P}, \quad \tilde{P} = -\tilde{M}\tilde{\Lambda} + \tilde{Q}.$$

To simplify notations we set $\Psi = \zeta^2(\chi^2 \log \phi + \Phi)$. Then we have

Lemma 3.4 *We have $T = e^{n\Psi} \in S(e^{n\Psi}, (\log^2 \langle \xi' \rangle_\mu) \bar{g})$.*

Proof: Note that $\partial_x^\beta \partial_\xi^\alpha \log \phi = \phi^{-1} \partial_x^\beta \partial_\xi^\alpha \phi$ and $\phi^{-1} \in S(\phi^{-1}, g)$ for $|\alpha + \beta| = 1$. Since $|\log \phi| \leq C \log \langle \xi' \rangle_\mu$ and $g, g_1 \leq \bar{g}$ the assertion is clear. \square

Let us write $\tilde{M} = D_0 - \tilde{m}(x, D')$, $\tilde{\Lambda} = D_0 - \tilde{\lambda}(x, D')$ and fix any small $\varepsilon > 0$.

Proposition 3.1 ([15, 16]) *Let $\tilde{P} = -(\tilde{M} - i\gamma \lambda_\mu^{2\varepsilon})(\tilde{\Lambda} - i\gamma \lambda_\mu^{2\varepsilon}) + \tilde{Q}$ then we have*

$$(3.12) \quad \begin{aligned} 2\text{Im}(\tilde{P}u, \tilde{\Lambda}u) &\geq \frac{d}{dx_0} (\|\tilde{\Lambda}u\|^2 + ((\text{Re } \tilde{Q})u, u) + \gamma^2 \|\langle D' \rangle_\mu^{2\varepsilon} u\|^2) \\ &\quad + \gamma \|\lambda_\mu^\varepsilon (\tilde{\Lambda}u)\|^2 + 2\gamma \text{Re}(\lambda_\mu^{2\varepsilon} (\tilde{Q}u), u) + 2((\text{Im } \tilde{m})\tilde{\Lambda}u, \tilde{\Lambda}u) \\ &\quad + 2\text{Re}(\tilde{\Lambda}u, (\text{Im } \tilde{Q})u) + \text{Im}([D_0 - \text{Re } \tilde{\lambda}, \text{Re } \tilde{Q}]u, u) \\ &\quad + 2\text{Re}((\text{Re } \tilde{Q})u, (\text{Im } \tilde{\lambda})u) + \frac{\gamma^3}{2} \|\lambda_\mu^{3\varepsilon} u\|^2 + 2\gamma^2 (\lambda_\mu^{4\varepsilon} (\text{Im } \tilde{\lambda})u, u). \end{aligned}$$

In this paper positive large parameters n, γ and a positive small parameter μ are assumed to satisfy $n\mu^{1/4} \ll 1$ and $\gamma\mu^4 \gg 1$.

Remark 3.5 The weight $\langle \mu D' \rangle^{2\varepsilon}$ is introduced to control error terms $\log^N \langle D' \rangle$, caused by metric $(\log^2 \langle \xi' \rangle_\mu) \bar{g}$, and hence we can choose $\varepsilon > 0$ as small as we please, which determines the well-posed Gevrey class $\gamma^{(1/2\varepsilon)}$. Actually the Cauchy problem is well-posed in the space consisting of all C_0^∞ functions with Fourier transform bounded by $\exp(-C \log^N \langle \xi' \rangle)$ with some $C > 0, N > 0$.

Definition 3.6 We set $\lambda = \langle \mu \xi' \rangle$, $\lambda_\mu = \langle \xi' \rangle_\mu$ and $\lambda^{s+0} = \langle \mu \xi' \rangle^s \langle \xi' \rangle_\mu^{+0}$. We write $a \in S(\lambda_\mu^{s+0}, g)$ ($a \in S(\lambda^{s+0}, g)$) if $a \in S(\langle \xi' \rangle_\mu^{s+\varepsilon}, g)$ ($a \in S(\langle \mu \xi' \rangle^s \langle \xi' \rangle_\mu^\varepsilon, g)$) for any $\varepsilon > 0$. We also denote

$$\|Au\| \leq C \|\lambda_\mu^{s+0} u\| \quad (\|\lambda^{s+0} u\|)$$

if $\|Au\| \leq C \|\langle D' \rangle_\mu^{s+\varepsilon} u\|$ ($\|Au\| \leq C \|\langle \mu D' \rangle^s \langle D' \rangle_\mu^\varepsilon u\|$) for any $\varepsilon > 0$ with some $C > 0$ independent of $\varepsilon > 0$.

4 Transformed symbols $\tilde{\lambda}$, \tilde{m}

We first list up several properties of cutoff symbols.

Lemma 4.1 *We have*

$$(4.1) \quad \begin{aligned} \chi \chi_2 &= 0, \quad \zeta \zeta_- = 0, \quad \zeta \zeta_+ = \zeta_+, \quad \tilde{\zeta} \zeta = \nu \zeta_+^2, \\ \hat{\phi}_2, \quad \chi \hat{\phi}_1^2, \quad \zeta' \hat{\theta}, \quad \zeta'_\pm \hat{\theta} &\in S(w, g), \quad \chi \hat{\phi}_1 \in S(w^{1/2}, g), \\ (1 - \zeta_-^2 - \zeta_+^2) \hat{\theta}, \quad \zeta(1 - \zeta_+^2) \hat{\theta} &\in S(w, g) \end{aligned}$$

where $\zeta' = \zeta'(\hat{\theta} w^{-1})$. We also have $\{\chi, \lambda_\mu^s\}, \{\zeta, \lambda_\mu^s\} \in S(w^{-1} \lambda_\mu^{s-1}, g)$.

Denote $W_\beta^\alpha = T^{-1} \partial_x^\beta \partial_{\xi'}^\alpha T$ and note that we have for $a \in S(\lambda_\mu^{s+0} w^t, g)$ or $a \in S(\lambda^s, g_0)$

$$\begin{aligned} a \# T &= T \# a - inT\{a, \Psi\} \\ &+ \frac{i}{8} T \sum_{|\alpha+\beta|=3} \frac{(-1)^{|\beta|}}{\alpha! \beta!} (a_{(\beta)}^{(\alpha)} W_\alpha^\beta - W_\beta^\alpha a_{(\alpha)}^{(\beta)}) + T \# R \end{aligned}$$

with some $R \in S(w^t \lambda_\mu^{s-5/4+0}, \bar{g})$ or $R \in S(\lambda^s \lambda_\mu^{-5/2+0}, \bar{g})$ respectively. From Lemma 3.3 it follows that $\psi_{(\beta)}^{(\alpha)} W_\alpha^\beta \in S(1, \bar{g})$ for $|\alpha+\beta|=3$ then the main parts of $\text{Im } \tilde{m}$ and $\text{Im } \tilde{\lambda}$ are, up to the parameter n

$$\begin{aligned} \{\xi_0 \pm \phi_1 \mp \psi, \Psi\} &= \zeta^2 \{\xi_0 \pm \phi_1 \mp \psi, \chi^2 \log \phi + \Phi\} \\ &+ \{\xi_0 \pm \phi_1 \mp \psi, \zeta^2\} (\chi^2 \log \phi + \Phi). \end{aligned}$$

To estimate $\{\xi_0 \pm \phi_1 \mp \psi, \chi^2 \log \phi + \Phi\}$ it suffices to repeat similar arguments as in [16] to get

$$(4.2) \quad \{\xi_0 \pm \phi_1 \mp \psi, \chi^2 \log \phi + \Phi\} = \{\xi_0 \pm \phi_1 \mp \psi, \hat{\phi}_2\} (r + 2\omega \rho^{-2}) + R$$

with $R \in S(\lambda_\mu^{+0}, \bar{g})$ where

$$\begin{aligned} 0 \leq r &= \chi^2 w^{-1} + \delta \in S(w^{-1} \lambda_\mu^{+0}, g), \\ 0 \leq \delta &= -2\chi \chi' \hat{\phi}_1^2 w^{-3} \hat{\phi}_2 \log \phi \in S(w^{-1} \lambda_\mu^{+0}, g) \end{aligned}$$

and the fact $\delta \geq 0$ follows from [16, Lemma 3.6] which was a key point to treat (4.2). We check how the term $\{\xi_0 \pm \phi_1 \mp \psi, \zeta^2\}(\chi^2 \log \phi + \Phi)$ can be managed. It is not difficult to see

$$(4.3) \quad \begin{aligned} & \{\xi_0 \pm \phi_1 \mp \psi, \zeta^2\}(\chi^2 \log \phi + \Phi) \\ &= -2\zeta\zeta'\hat{\theta}w^{-3}\hat{\phi}_2(\chi^2 \log \phi + \Phi)\{\xi_0 \pm \phi_1 \mp \psi, \hat{\phi}_2\} + R \end{aligned}$$

with $R \in \mu S(\lambda_\mu^{+0}, \bar{g})$. Here we note

Lemma 4.2 *We have*

$$0 \leq \Delta = -2\zeta\zeta'\hat{\theta}w^{-3}\hat{\phi}_2(\chi^2 \log \phi + \Phi) \in S(w^{-1}\lambda_\mu^{+0}, \bar{g}).$$

Proof: Since $\hat{\phi}_2 \log \phi \geq 0$ by [17, Lemma 3.6] it is clear $0 \leq -2\chi^2\zeta\zeta'\hat{\theta}w^{-3}\hat{\phi}_2 \log \phi \in S(w^{-1}\lambda_\mu^{+0}, \bar{g})$ because $\zeta'(\hat{\theta}w^{-1})\hat{\theta} \leq 0$. Noting that $0 \leq \Phi = \pi - 2 \arg(\hat{\phi}_2 + i\omega) \leq \pi$ if $\hat{\phi}_2 \geq 0$ and $-\pi \leq \Phi = \pi - 2 \arg(\hat{\phi}_2 + i\omega) \leq 0$ for $\hat{\phi}_2 \leq 0$ it is also clear $\hat{\phi}_2\Phi \geq 0$ and hence $0 \leq -2\zeta\zeta'\hat{\theta}w^{-3}\hat{\phi}_2\Phi \in S(w^{-1}, \bar{g})$. Thus we get the assertion. \square

To simplify notations we set $\Gamma = r + 2\omega\rho^{-2}$. From (4.2) and (4.3) it suffices to consider $n(\Delta + \zeta^2\Gamma)\{\xi_0 \pm \phi_1 \mp \psi, \hat{\phi}_2\}$. As in [16] we set

$$\begin{cases} e_1 = \mu\hat{c} + \nu\{\phi_1, \hat{\phi}_2\}, & e_3 = \{\xi_0 + \phi_1, \hat{\phi}_2\}, \\ e_2 = \{\xi_0 + \phi_1, \hat{\phi}_2\} - \nu\hat{\theta}\{\phi_1, \hat{\phi}_2\}\zeta_+^2. \end{cases}$$

Noting Lemma 4.1 it is easy to see

$$(4.4) \quad \begin{aligned} \{\xi_0 - \phi_1 + \psi, \hat{\phi}_2\} &= \mu\hat{c}\hat{\theta} + \tilde{\zeta}\{\phi_1, \hat{\phi}_2\}\hat{\theta} + c_0\hat{\theta}\hat{\phi}_1 + 3\chi_2\hat{\phi}_1^2\{\phi_1, \hat{\phi}_2\}, \\ \{\xi_0 + \phi_1 - \psi, \hat{\phi}_2\} &= \{\xi_0 + \phi_1, \hat{\phi}_2\} - \tilde{\zeta}\{\phi_1, \hat{\phi}_2\}\hat{\theta} - 3\chi_2\hat{\phi}_1^2\{\phi_1, \hat{\phi}_2\} \end{aligned}$$

modulo $S(w, \bar{g})$. Noting $\zeta = \zeta_+^2 + \zeta(1 - \zeta_+^2)$, $\zeta(1 - \zeta_+^2)\hat{\theta} \in S(w, \bar{g})$ we see

$$\begin{cases} \zeta^2\{\xi_0 - \phi_1 + \psi, \hat{\phi}_2\} = (e_1 + a_1\hat{\phi}_1)\zeta_+^2\hat{\theta} + a_2\zeta\hat{\phi}_1^2, \\ \zeta^2\{\xi_0 + \phi_1 - \psi, \hat{\phi}_2\} = e_2\zeta^2 + a_3\zeta\hat{\phi}_1^2 \end{cases}$$

with $a_i \in S(1, \bar{g})$ modulo $S(w, \bar{g})$. Since $\Delta\hat{\theta}, \Gamma\hat{\phi}_1^2 \in S(\lambda_\mu^{+0}, \bar{g})$ by Lemma 4.1 we see $\text{Im } \tilde{\lambda} = n\zeta_+^2\Gamma(e_1 + a\hat{\phi}_1) + R$ with $R \in S(\lambda_\mu^{+0}, \bar{g})$ and $a \in S(\lambda_\mu^{+0}, \bar{g})$. Similarly we have $\text{Im } \tilde{m} = n\Delta(e_3 + a'\hat{\phi}_1^2) + n\zeta^2\Gamma e_2 + R$ with $R \in S(\lambda_\mu^{+0}, \bar{g})$. Noting that the main part of $\text{Re } \tilde{\lambda}$ comes from $\{\{\xi_0 - \phi + \psi, \Psi\}, \Psi\}$ we summarize

Lemma 4.3 *We have*

$$\begin{cases} \text{Im } \tilde{\lambda} = n(e_1 + b_1\hat{\phi}_1)\Gamma\zeta_+^2\hat{\theta} + R_1, \\ \text{Re } \tilde{\lambda} = \phi_1 - \psi + n(b_2\hat{\theta} + b_3\hat{\phi}_1^2)w^{-1/2} + R_2, \\ \text{Im } \tilde{m} = ne_2\zeta^2\Gamma + n(e_3 + b_4\hat{\phi}_1^2)\Delta + R_3 \end{cases}$$

where $b_i, R_i \in S(\lambda_\mu^{+0}, \bar{g})$.

Lemma 4.4 *There exists $c > 0$ which is independent of $\nu > 0$ such that we have*

$$\begin{aligned} C(\operatorname{Im} \tilde{\lambda} u, u) &\geq c\mu n(\Gamma \zeta_+^2 \hat{\theta} u, u) - C_1 \|\lambda_\mu^{+0} u\|^2 \\ &\geq c\mu n(\Gamma(\zeta_+ \hat{\theta}^{1/2})u, (\zeta_+ \hat{\theta}^{1/2})u) - C_2 \|\lambda_\mu^{+0} u\|^2, \\ C(\operatorname{Im} \tilde{m} u, u) &\geq c\mu n((\zeta^2 \Gamma + \Delta)u, u) - C_4 \|\lambda_\mu^{+0} u\|^2 \\ &\geq c\mu n(\Gamma(\zeta u), \zeta u) + c\mu n(\Delta u, u) - C_5 \|\lambda_\mu^{+0} u\|^2. \end{aligned}$$

We have also

$$\begin{aligned} C(\operatorname{Im} \tilde{\lambda} u, u) &\geq c\mu n(\|\chi \zeta_+ \hat{\theta}^{1/2} w^{-1/2} u\|^2 + \|\zeta_+ \hat{\theta}^{1/2} \rho^{-1/2} u\|^2), \\ C(\operatorname{Im} \tilde{m} u, u) &\geq c\mu n(\|\zeta \chi w^{-1/2} u\|^2 + \|\zeta \rho^{-1/2} u\|^2) \end{aligned}$$

modulo $C\|\lambda_\mu^{+0} u\|^2$ with some $C, C' > 0$ independent of μ .

Proof: Since $\hat{\phi}_1(0, \mathbf{e}_2) = 0$ we may assume $\tilde{e}_1 = e_1 + b_1 \hat{\phi}_1 \geq \mu c_1 > 0$. Take $M > 0$ so that $M\tilde{e}_1 \geq \mu$. Since $0 \leq (M\tilde{e}_1 - \mu)\Gamma \zeta_+^2 \hat{\theta} \in \mu S(w^{-1}\lambda_\mu^{+0}, \bar{g}) \subset \mu S_{3/4, 1/2}^{1/2+0}$ then from the Fefferman-Phong inequality (see [7, Theorem 18.6.8]) it follows that

$$M(\tilde{e}_1 \Gamma \zeta_+^2 \hat{\theta} u, u) \geq \mu(\Gamma \zeta_+^2 \hat{\theta} u, u) - C_1 \|\lambda_\mu^{+0} u\|^2.$$

Here note that $\Gamma \zeta_+^2 \hat{\theta} = (\zeta_+ \hat{\theta}^{1/2}) \# \Gamma \# (\zeta_+ \hat{\theta}^{1/2}) + R$ with $R \in S(\lambda_\mu^{+0}, \bar{g})$. Since $|(Ru, u)| \leq C' \|\lambda_\mu^{+0} u\|^2$ for $R \in S(\lambda_\mu^{+0}, \bar{g})$ the first assertion follows. To show the second assertion it suffices to repeat the same arguments proving the first assertion.

To prove the third assertion we first note that

$$(\delta \zeta_+^2 \hat{\theta} u, u), (\zeta^2 \delta u, u), (\Delta u, u) \geq -C \|\lambda_\mu^{+0} u\|^2$$

which follows the Fefferman-Phong inequality since $\delta, \Delta \in S_{3/4, 1/2}^{1/2+0}$ are non-negative. We then write $\chi^2 \zeta_+^2 \hat{\theta} w^{-1} = \chi \zeta_+ \hat{\theta}^{1/2} w^{-1/2} \# \chi \zeta_+ \hat{\theta}^{1/2} w^{-1/2} + R$ with $R \in S(\lambda_\mu^{+0}, \bar{g})$ because $\zeta_+ \hat{\theta}^{1/2} \in S(1, G) \subset S(1, \bar{g})$ by Lemma 3.1 which gives the first term on the right-hand side. To get the second term on the right-hand side we note that on the support of $1 - \chi^2$ we have $C\omega \geq \rho$ with some $C > 0$ and it is obvious that $w^{-1} \geq \rho^{-1}$. Therefore it follows $C(\chi^2 \zeta_+^2 \hat{\theta} w^{-1} + \zeta_+^2 \hat{\theta} \omega \rho^{-2}) \geq \zeta_+^2 \rho^{-1} \hat{\theta}$. Then the Fefferman-Phong inequality gives

$$C(\chi^2 \zeta_+^2 \hat{\theta} w^{-1} u, u) + C(\zeta_+^2 \hat{\theta} \omega \rho^{-2} u, u) \geq \|\zeta_+ \rho^{-1/2} \hat{\theta} u\|^2 - C \|\lambda_\mu^{+0} u\|^2$$

which gives the second term. The proof of the last assertion is similar. \square

Applying Lemma 4.4 one can show

Proposition 4.1 *We have*

$$\begin{aligned} 2((\operatorname{Im} \tilde{m}) \tilde{\Lambda} u, \tilde{\Lambda} u) &\geq c\mu n((\Gamma + \Delta)(\zeta \tilde{\Lambda} u), (\zeta \tilde{\Lambda} u)) + c\mu n\|\chi \zeta w^{-1/2} \tilde{\Lambda} u\|^2 \\ &\quad + c\mu n\|\zeta \omega^{1/2} \rho^{-1} \tilde{\Lambda} u\|^2 + c\mu n\|\zeta \rho^{-1/2} \tilde{\Lambda} u\|^2 - C \|\lambda_\mu^{+0} \tilde{\Lambda} u\|^2 \end{aligned}$$

with some $c > 0$ independent of $\nu > 0$ and some $C > 0$.

5 Estimate $\|\tilde{\Lambda}u\|$

We first remark the following lemma which is easily checked using (3.1) and (3.2).

Lemma 5.1 *Let $\hat{\zeta}, \hat{\chi} \in C^\infty(\mathbb{R})$ such that $\hat{\zeta}', \hat{\chi}' \in C_0^\infty(\mathbb{R})$. Set $\hat{\zeta} = \hat{\zeta}(\hat{\theta}w^{-1})$ and $\hat{\chi} = \hat{\chi}(\hat{\phi}_1^2w^{-1})$. Then we have*

$$\begin{aligned} \{\hat{\phi}_1, \hat{\zeta}\} &\in S(w^{-1}\lambda_\mu^{-1}, \bar{g}), \{\hat{\phi}_1, \hat{\chi}\} \in S(w^{-1}\lambda_\mu^{-1}, \bar{g}), \\ \{\hat{\phi}_1, w^{-1}\} &\in S(w^{-2}\lambda_\mu^{-1}, \bar{g}), \{\hat{\phi}_1, \omega\rho^{-2}\} \in S(\rho^{-2}\lambda_\mu^{-1}, \bar{g}), \\ \{\hat{\phi}_2, \hat{\chi}\} &\in S(w^{-1/2}\lambda_\mu^{-1}, \bar{g}), \{\hat{\phi}_2, \hat{\zeta}\} \in S(\lambda_\mu^{-1}, \bar{g}), \\ \{\hat{\phi}_2, \omega\rho^{-2}\} &\in S(\rho^{-3/2}\lambda_\mu^{-1}, \bar{g}), \{\hat{\phi}_2, w^{-1}\} \in S(w^{-1}\lambda_\mu^{-1}, \bar{g}), \\ \{\hat{\zeta}, \hat{\theta}\} &\in S(\lambda_\mu^{-1}, \bar{g}), \{\hat{\zeta}, w^{-1}\} \in S(w^{-2}\lambda_\mu^{-1}, \bar{g}), \{\hat{\zeta}, \omega\rho^{-2}\} \in S(\rho^{-3/2}w^{-1}\lambda_\mu^{-1}, \bar{g}), \\ \{\hat{\chi}, \hat{\theta}\} &\in S(\lambda_\mu^{-1}, \bar{g}), \{\hat{\chi}, w^{-1}\} \in S(w^{-1/2}, \bar{g}), \{\hat{\chi}, \omega\rho^{-2}\} \in S(\rho^{-5/2}\lambda_\mu^{-1}, \bar{g}), \\ \{\hat{\zeta}, \hat{\chi}\} &\in S(w^{-3/2}\lambda_\mu^{-1}, \bar{g}), \{w^{-1}, \omega\rho^{-2}\} \in S(w^{-2}\rho^{-3/2}\lambda_\mu^{-1}, \bar{g}), \\ \{\hat{\theta}, w^{-1}\} &\in S(w^{-1}\lambda_\mu^{-1}, \bar{g}), \{\hat{\theta}, \omega\rho^{-2}\} \in S(\rho^{-1}\lambda_\mu^{-1}, \bar{g}), \{w^{-1}, \lambda_\mu^{-1}\} \in S(\lambda_\mu^{-1}, \bar{g}). \end{aligned}$$

From Lemma 4.4 it follows that

$$(5.1) \quad \begin{aligned} -2\text{Im}(\tilde{\Lambda}v, v) &\geq \frac{d}{dx_0}\|v\|^2 + \frac{3}{2}\gamma\|\lambda_\mu^\varepsilon v\|^2 + c\mu n(\chi^2\zeta_+^2\hat{\theta}w^{-1}v, v) \\ &\quad + c\mu n\|\chi\zeta_+\hat{\theta}^{1/2}w^{-1/2}v\|^2 - C\|\lambda_\mu^{+0}v\|^2 \end{aligned}$$

with some $c > 0$. Let $\zeta_0(s), \chi_0(s) \in C^\infty(\mathbb{R})$ be such that $\text{supp } \zeta_0$ is contained in $\{\zeta_+ = 1\}$ and $\chi_0 = 1$ for $s \leq c$ with some $c > 0$ and $\text{supp } \chi_0 \subset \{\chi = 1\}$. Set $\zeta_0 = \zeta_0(\hat{\theta}w^{-1})$ and $\chi_0 = \chi_0(\hat{\phi}_1^2w^{-1})$. Replace u by $w^{-1}\eta\hat{\theta}^{1/2}u$, $\eta = \chi_0\zeta_0$ in (5.1) it follows that

$$(5.2) \quad \begin{aligned} -2\text{Im}(\tilde{\Lambda}(w^{-1}\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u) &\geq \frac{d}{dx_0}\|w^{-1}\eta\hat{\theta}^{1/2}u\|^2 \\ &\quad + \frac{\gamma}{2}\|\lambda^\varepsilon w^{-1}\eta\hat{\theta}^{1/2}u\|^2 + c\mu n(w^{-1}\chi^2\zeta_+^2\hat{\theta}(w^{-1}\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u). \end{aligned}$$

We first examine $[\tilde{\Lambda}, w^{-1}\eta\hat{\theta}^{1/2}]u$. Note $\{\xi_0 - \phi_1, w^{-1}\} = -2\hat{\phi}_2w^{-3}\{\xi_0 - \phi_1, \hat{\phi}_2\}$ modulo $S(w^{-1}, \bar{g})$. From (3.1) we see $\eta\hat{\theta}^{1/2}\{\xi_0 - \phi_1, w^{-1}\} - c\eta w^{-2}\hat{\theta}^{3/2} \in S(w^{-1}, \bar{g})$ with some $c \in S(1, \bar{g})$. Since $\eta\hat{\theta}^{1/2}\{\psi, w^{-1}\} = \eta\hat{\theta}^{1/2}\{\tilde{\zeta}\hat{\theta}\phi_1, w^{-1}\}$ for $\chi\chi_2 = 0$ then noting Lemma 5.1 we have $\eta\hat{\theta}^{1/2}\{\xi_0 - \phi_1 + \psi, w^{-1}\} - b\eta w^{-2}\hat{\theta}^{3/2} \in S(w^{-1}, \bar{g})$. We next examine $w^{-1}\{\xi_0 - \phi_1 + \psi, \eta\hat{\theta}^{1/2}\}$. Since $\hat{\theta}^{-1/2}\zeta_0 \in S(w^{-1/2}, \bar{g})$ from (3.1) we have $w^{-1}\{\xi_0 - \phi_1, \eta\hat{\theta}^{1/2}\} - c\eta\hat{\theta}w^{-3/2} \in \mu S(w^{-1}, \bar{g})$ by similar arguments. Noting $\{\tilde{\zeta}\hat{\theta}\phi_1, \eta\hat{\theta}^{1/2}\} \in \mu S(w^{-1}, \bar{g})$ we get

$$(5.3) \quad \{\tilde{\Lambda}, w^{-1}\eta\hat{\theta}^{1/2}\} - c\eta w^{-2}\hat{\theta}^{3/2} - c'\eta\hat{\theta}w^{-3/2} \in \mu S(w^{-1}, \bar{g})$$

with some $c, c' \in \mu S(1, \bar{g})$. From (5.3) one has

$$\begin{aligned} |\operatorname{Im}([\tilde{\Lambda}, w^{-1}\eta\hat{\theta}^{1/2}]u, w^{-1}\eta\hat{\theta}^{1/2}u)| &\leq \operatorname{Re}(c\eta w^{-2}\hat{\theta}^{3/2}u, w^{-1}\eta\hat{\theta}^{1/2}u) \\ &\quad + \operatorname{Re}(c'\eta\hat{\theta}w^{-3/2}u, w^{-1}\eta\hat{\theta}^{1/2}u) + C\|w^{-1}u\|^2. \end{aligned}$$

Write $\operatorname{Re}(w^{-1}\eta\hat{\theta}^{1/2}\#c\eta w^{-2}\hat{\theta}^{3/2}) = \operatorname{Re}(w^{-3/2}\eta\hat{\theta}\#c\# \eta w^{-3/2}\hat{\theta})$ modulo $S(w^{-2}, \bar{g})$ and $\tilde{\Lambda}(w^{-1}\eta\hat{\theta}^{1/2}u) = w^{-1}\eta\hat{\theta}^{1/2}(\tilde{\Lambda}u) + [\tilde{\Lambda}, w^{-1}\eta\hat{\theta}^{1/2}]u$ we get

$$(5.4) \quad \begin{aligned} \operatorname{Im}(\tilde{\Lambda}w^{-1}(\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u) &\leq \operatorname{Im}(w^{-1}\eta\hat{\theta}^{1/2}(\tilde{\Lambda}u), w^{-1}\eta\hat{\theta}^{1/2}u) \\ &\quad + C\mu\|\eta w^{-3/2}\hat{\theta}u\|^2 + C\|\eta w^{-1}\hat{\theta}^{1/2}u\|^2 + C\|w^{-1}u\|^2. \end{aligned}$$

We now estimate $\operatorname{Im}(w^{-1}\eta\hat{\theta}^{1/2}(\tilde{\Lambda}u), w^{-1}\eta\hat{\theta}^{1/2}u)$. Thanks to Lemma 3.3 one can write

$$w^{-1}\eta\hat{\theta}^{1/2}\#w^{-1}\eta\hat{\theta}^{1/2} = \eta w^{-1/2}\#\eta w^{-3/2}\hat{\theta} + b\zeta_0 w^{-3/2}\hat{\theta} + R$$

with $b \in S(1, \bar{g})$ where $R \in S(w^{-1}, \bar{g})$ and therefore we have

$$(5.5) \quad \begin{aligned} \operatorname{Im}(w^{-1}\eta\hat{\theta}^{1/2}(\tilde{\Lambda}u), w^{-1}\eta\hat{\theta}^{1/2}u) &\leq \operatorname{Im}(w^{-1/2}\eta(\tilde{\Lambda}u), \eta w^{-3/2}\hat{\theta}u) \\ &\quad + C(\|\tilde{\Lambda}u\|^2 + \|\zeta_0 w^{-3/2}\hat{\theta}u\|^2) + C\|w^{-1}u\|^2 + C\|u\|^2 \\ &\leq (c\mu n)^{-1}\|\eta w^{-1/2}\tilde{\Lambda}u\|^2 + (c\mu n/2)\|\zeta_0 w^{-3/2}\hat{\theta}u\|^2 \\ &\quad + C(\|\tilde{\Lambda}u\|^2 + \|w^{-1}u\|^2) \end{aligned}$$

where $b \in S(1, \bar{g})$. Combining (5.2), (5.4) and (5.5) one obtains

$$(5.6) \quad \begin{aligned} &\frac{d}{dx_0}\|w^{-1}\eta\hat{\theta}^{1/2}u\|^2 + \frac{\gamma}{2}\|\lambda^\varepsilon w^{-1}\eta\hat{\theta}^{1/2}u\|^2 \\ &\quad + c\mu n(w^{-1}\chi^2\zeta_+^2\hat{\theta}(w^{-1}\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u) \\ &\leq C(\mu n)^{-1}\|\eta w^{-1/2}\tilde{\Lambda}u\|^2 + c\mu n\|\zeta_0 w^{-3/2}\hat{\theta}u\|^2 \\ &\quad + C(\|\eta w^{-1}\hat{\theta}^{1/2}u\|^2 + \|w^{-1}u\|^2 + \|\tilde{\Lambda}u\|^2). \end{aligned}$$

We now estimate $(w^{-1}\chi^2\zeta_+^2\hat{\theta}(w^{-1}\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u)$ from below. Note that

$$w^{-1}\eta\hat{\theta}^{1/2}\#w^{-1}\chi^2\zeta_+^2\hat{\theta}\#w^{-1}\eta\hat{\theta}^{1/2} = w^{-3}\eta^2\chi^2\zeta_+^2\hat{\theta}^2 + R$$

with $R \in S(w^{-2}, \bar{g})$ and hence we have

$$(w^{-1}\chi^2\zeta_+^2\hat{\theta}(w^{-1}\eta\hat{\theta}^{1/2}u), w^{-1}\eta\hat{\theta}^{1/2}u) \geq (w^{-3}\eta^2\hat{\theta}^2u, u) - C\|w^{-1}u\|^2$$

for $\eta^2\chi^2\zeta_+^2 = \eta^2$. Since $\eta^2w^{-3}\hat{\theta}^2 = \eta w^{-3/2}\hat{\theta}\#\eta w^{-3/2}\hat{\theta} + R$ with $R \in S(w^{-2}, \bar{g})$ which proves that

$$(5.7) \quad \begin{aligned} 2\operatorname{Re}(w^{-1}\chi^2\zeta_+^2\hat{\theta}(\eta w^{-1}\hat{\theta}^{1/2}u), \eta w^{-1}\hat{\theta}^{1/2}u) &\geq (\eta^2w^{-3}\hat{\theta}^2u, u) \\ &\quad + \|\eta w^{-3/2}\hat{\theta}u\|^2 - C\|w^{-1}u\|^2. \end{aligned}$$

Write $\zeta_0^2 w^{-3} \hat{\theta}^2 = \eta^2 w^{-3} \hat{\theta}^2 + (1 - \chi_0^2) \zeta_0^2 w^{-3} \hat{\theta}^2$ and consider

$$\mu^{-2} M \zeta_0^2 \hat{\theta} \phi_1^2 - (1 - \chi_0^2) \zeta_0^2 w^{-3} \hat{\theta}^2 = (\mu^{-1} H \zeta_0 \hat{\theta}^{1/2} \phi_1)^2$$

where $H = (M - (1 - \chi_0^2) w^{-3} \lambda_\mu^{-2} \hat{\phi}_1^{-2} \hat{\theta})^{1/2} \in S(1, \bar{g})$ for large $M > 0$ which follows from $(1 - \chi_0^2) \hat{\phi}_1^{-2} \in S(w^{-1}, \bar{g})$. Then it is not difficult to see

$$\begin{aligned} (\mu^{-1} H \zeta_0 \hat{\theta}^{1/2} \phi_1) \# (\mu^{-1} H \zeta_0 \hat{\theta}^{1/2} \phi_1) &= (\mu^{-1} H \zeta_0 \hat{\theta}^{1/2} \phi_1)^2 \\ &\quad + b_1 w^{-1/2} \phi_1 + b_2 w^{-1} \hat{\phi}_1^2 \lambda + R \end{aligned}$$

with $b \in \mu^{-1} S(1, \bar{g})$ and $R \in S(w^{-2}, \bar{g})$. Noting $b_1 w^{-1/2} \phi_1 = w^{-1} \# b_1 w^{1/2} \phi_1 + R_1$ and $b_2 w^{-1} \hat{\phi}_1^2 \lambda = w^{-1} \# b_2 \hat{\phi}_1^2 \lambda + R_2$ with $R_i \in S(w^{-2}, \bar{g})$ we conclude that $\mu^{-2} M (\zeta_0^2 \hat{\theta} \phi_1^2 u, u) \geq ((1 - \chi_0^2) \zeta_0^2 w^{-3} \hat{\theta}^2 u, u)$ modulo a term $C \mu^{-2} (\|b_1 w^{1/2} \phi_1 u\|^2 + \|b_2 \hat{\phi}_1^2 \lambda u\|^2 + \|w^{-1} u\|^2)$ which proves together with (5.7)

$$\begin{aligned} |(\zeta_0^2 w^{-3} \hat{\theta}^2 u, u)| &\leq C \mu^{-2} (a^2 \tilde{\zeta} \hat{\theta} \phi_1^2 u, u) + 2(w^{-1} \chi^2 \zeta_+^2 \hat{\theta} (\eta w^{-1} \hat{\theta}^{1/2}) u, \eta w^{-1} \hat{\theta}^{1/2} u) \\ &\quad + C \mu^{-2} (\|w^{1/2} \phi_1 u\|^2 + \|\hat{\phi}_1^2 \lambda u\|^2 + \|w^{-1} u\|^2) \end{aligned}$$

with some $C > 0$. To simplify notations we introduce

Definition 5.2 We denote by $O(E)$ a symbol or the set of symbols of the form

$$\begin{aligned} a_1 \mu w^{-1} + a_2 \mu \omega^{-1} + a_3 \mu^{1/2} \lambda^{1/2} + a_4 w^{1/2} \phi_1 \\ + a_5 \omega^{1/2} \phi_1 + a_6 \phi_2 + a_7 \hat{\phi}_1^2 \lambda + a_8 w \lambda + a_9 \omega \lambda \end{aligned}$$

with $a_i \in S(\lambda_\mu^{+0}, \bar{g})$. We denote $S(\lambda^{t_1} \lambda_\mu^{t_2} w^s, \bar{g}) O(E)$ a symbol or the set of symbols which is a linear combination of w^{-1} , ω^{-1} , $\lambda^{1/2}$, $w^{1/2} \phi_1$, $\omega^{1/2} \phi_1$, ϕ_2 , $\lambda \hat{\phi}_1^2$, $w \lambda$ and $\omega \lambda$ with coefficients in $S(\lambda^{t_1} \lambda_\mu^{t_2} w^s, \bar{g})$. We also denote

$$\begin{aligned} \|O(E)u\|^2 &= \mu^2 (\|w^{-1} u\|^2 + \|\omega^{-1} u\|^2) + \mu \|\lambda^{1/2} u\|^2 + \|w^{1/2} \phi_1 u\|^2 \\ &\quad + \|\omega^{1/2} \phi_1 u\|^2 + \|\phi_2 u\|^2 + \|\hat{\phi}_1^2 \lambda u\|^2 + \|w \lambda u\|^2 + \|\omega \lambda u\|^2. \end{aligned}$$

Proposition 5.1 Let χ_0, ζ_0 be as above. Then we have

$$\begin{aligned} C \mu n \|\chi_0 \zeta_0 w^{-1/2} \tilde{\Lambda} u\|^2 + C \mu^2 n^2 \|\tilde{\Lambda} u\|^2 &\geq c \mu^2 n^2 \frac{d}{dx_0} \|\chi_0 \zeta_0 w^{-1} \hat{\theta}^{1/2} u\|^2 \\ + c \mu^2 n^2 \|\chi_0 \zeta_0 w^{-1} \hat{\theta}^{1/2} \lambda^\varepsilon u\|^2 + c \mu^3 n^3 (\|\zeta_0 w^{-3/2} \hat{\theta} u\|^2 &+ (\zeta_0^2 w^{-3} \hat{\theta}^2 u, u)) \\ - C \mu (\tilde{\zeta} a^2 \hat{\theta} \phi_1^2 u, u) - C \mu \|O(E)u\|^2 \end{aligned}$$

with some $c > 0$ and $C = C(n)$.

Replacing u now by $w^{-1/2} \eta \hat{\theta}^{1/2} u$ in (5.1) and repeating similar arguments we obtain

Proposition 5.2 Let χ_0, ζ_0 be as above. Then we have

$$\begin{aligned} C \|\tilde{\Lambda} u\|^2 &\geq c \mu n \frac{d}{dx_0} \|\chi_0 \zeta_0 w^{-1/2} \hat{\theta}^{1/2} u\|^2 + c \mu n \gamma (\|\chi_0 \zeta_0 w^{-1/2} \hat{\theta}^{1/2} \lambda^\varepsilon u\|^2 \\ + c \mu^2 n^2 (\|\zeta_0 w^{-1} \hat{\theta} u\|^2 &+ (\zeta_0^2 w^{-2} \hat{\theta}^2 u, u)) - C \gamma^{1/2} \|O(E)u\|^2. \end{aligned}$$

6 Transformed symbol \tilde{Q}

We start with

Lemma 6.1 *One can write $O(E) = T\#(O(E) + R)$ with $R \in S(\lambda_\mu^{+0}, \bar{g})$.*

Proof: Let $A \in O(E)$. Then it is easy to check that $T_{(\beta)}^{(\alpha)} A_{(\alpha)}^{(\beta)} \in S(\lambda_\mu^{-1/4}, \bar{g})O(E)$ for $|\alpha + \beta| = 1$. Then we have $TA - T\#A = TA_1$ with $A_1 \in S(\lambda_\mu^{-1/4}, \bar{g})O(E)$. Repeating the same arguments we get $TA = T\#(A + A_1 + \dots + A_4) + K$ where $K \in S(\lambda_\mu^{-1}, \bar{g})O(E) \subset S(\lambda_\mu^{+0}, \bar{g})$. Since $T\#T^{-1} = 1 - r$ with $r \in \mu^{1/4}S(1, \bar{g})$ and hence the inverse of $1 - r$ exists in $\mathcal{L}(L^2, L^2)$ which is given by $Op(b)$ with $b \in S(1, \bar{g})$ (see [2]) and hence $T\#\tilde{T} = 1$ with $\tilde{T} = T^{-1}\#b \in S(\lambda_\mu^{+0}, \bar{g})$. Then writing $K = T\#(\tilde{T}\#K)$ we get the assertion. \square

Recall $W_\beta^\alpha = T^{-1}\partial_x^\beta \partial_\xi^\alpha T \in S(\lambda_\mu^{-3|\alpha|/4 + |\beta|/2 + 0}, \bar{g})$. Since $q_{(\beta)}^{(\alpha)} \in S(\lambda^2 \lambda_\mu^{-|\alpha|}, \bar{g})$ for $|\alpha + \beta| = 1$ by Lemma 3.3 we see

$$(6.1) \quad q\#T = T\#q - inT\{q, \Psi\} + \frac{i}{8}T \sum_{|\alpha+\beta|=3} \frac{(-1)^{|\beta|}}{\alpha!\beta!} (q_{(\beta)}^{(\alpha)} W_\alpha^\beta - W_\beta^\alpha q_{(\alpha)}^{(\beta)})$$

modulo $R \in \mu^{3/2}S(\lambda^{1/2+0}, \bar{g})$. We first check

Lemma 6.2 *We have*

$$\sum_{|\alpha+\beta|=3} (-1)^{|\beta|} (q_{(\beta)}^{(\alpha)} W_\alpha^\beta - W_\beta^\alpha q_{(\alpha)}^{(\beta)}) / \alpha!\beta! \in \mu S(\lambda_\mu^{+0}, \bar{g})O(E).$$

Proof: Write $q = \phi_2^2 + f\phi_1^2$ with $f = 2a^2\tilde{\zeta}\hat{\theta} + 2a^2\chi_2\hat{\phi}_1^2$ and recall $f \in S(1, g)$ and $f_{(\beta)}^{(\alpha)} \in S(\lambda_\mu^{-|\alpha|}, g)$ for $|\alpha + \beta| = 1$ by Lemma 3.3. Applying Lemma 3.3 again one can check that $(f\phi_1^2)_{(\beta)}^{(\alpha)} W_\alpha^\beta$ with $|\alpha + \beta| = 3$ is a linear combination of $\lambda^{1/2}$, $w^{1/2}\phi_1$ and $\hat{\phi}_1^2\lambda$ with coefficients in $\mu S(\lambda_\mu^{+0}, \bar{g})$ which proves the assertion. \square

We make more detailed studies on $\{q, \Psi\}$ and $\{\Psi, \{q, \Psi\}\}$. Let us denote $\Psi_1 = \zeta^2\chi^2 \log \phi \in S(\lambda_\mu^{+0}, g)$ and $\Psi_2 = \zeta^2\Phi$, $\Phi \in S(1, g_1)$ so that $\Psi = \Psi_1 + \Psi_2$.

Lemma 6.3 *We have*

$$\{q, \Psi\} = \nu\zeta_+^2 a^2 \hat{\theta} \phi_1 (\Gamma + \Delta) \{\phi_1, \hat{\phi}_2\} + a_1 O(E) + a_2 O(E) + a_3 O(E)$$

where $a_1 = \zeta\chi a'_1$ with $a'_1 \in \mu S(w^{-1/2}\lambda_\mu^{+0}, \bar{g})$ and $a_2 = \zeta a'_2$ with $a'_2 \in \mu S(\rho^{-1/2}, \bar{g})$ and $a_3 \in \mu S(\lambda_\mu^{+0}, \bar{g})$.

Proof: Thanks to (3.4) and Lemma 5.1 we can see $\{\phi_2^2 + a^2\chi_2\hat{\phi}_1^2\lambda^2, \Psi_2\} = a_2 O(E)$ where $a_2 = \zeta a'_2$ with $a_2 \in \mu S(\rho^{-1/2}, \bar{g})$. Similarly from Lemma 5.1 and

$$(6.2) \quad \{F, \log \phi\} = \{F, \hat{\phi}_2\}/w + \{F, \lambda_\mu^{-1}\}/(2w\phi)$$

we obtain $\{\phi_2^2 + a^2\chi_2\hat{\phi}_1^4\lambda^2, \Psi_1\} = a_1O(E)$ with $a_1 \in \mu S(w^{-1/2}\lambda_\mu^{+0}, \bar{g})$ where clearly $a_1 = \zeta\chi a'_1$. We turn to $\{a^2\tilde{\zeta}\hat{\theta}\phi_1^2, \Psi_j\}$. Repeating similar arguments one can check that $\{a^2\tilde{\zeta}\hat{\theta}\phi_1^2, \Psi_j\} = a^2\tilde{\zeta}\hat{\theta}\{\phi_1^2, \Psi_j\} + a_jO(E)$ where a_j verifies the same properties as above. From the same arguments proving (4.2) and (4.3) one can show

$$\{\phi_1, \Psi\} = (\zeta^2\Gamma + \Delta)\{\phi_1, \hat{\phi}_2\} + a_3O(E)$$

with $a_3 \in \mu S(\lambda_\mu^{+0}, \bar{g})$. Since $\tilde{\zeta}\zeta = \nu\zeta_+^2$ and $\tilde{\zeta}\Delta = \nu\zeta_+^2\Delta$ we get the assertion. \square

Lemma 6.4 *We have*

$$\{\Psi, \{q, \Psi\}\} = \zeta_+(a_1\mu w^{-2}\hat{\theta} + a_2w^{-1/2}\hat{\theta}\phi_1 + a_3\phi_1) + \mu S(\lambda_\mu^{+0}, \bar{g})O(E)$$

where $a_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$.

Proof: By obvious abbreviated notation we see $\Psi_{(\beta)}^{(\alpha)}(a_jO(E))_{(\alpha)}^{(\beta)} \in O(E)$ for $|\alpha + \beta| = 1$ and hence $\{\Psi, \sum a_jO(E)\} \in O(E)$. With $b = \zeta_+^2 a^2\hat{\theta}\Delta\{\phi_1, \hat{\phi}_2\} \in \mu S(\lambda_\mu^{+0}, \bar{g})$ noting $\hat{\theta}\Delta \in S(\lambda_\mu^{+0}, \bar{g})$ it is easy to check that $\{\Psi, b\phi_1\}$ is a linear combination of $w^{1/2}\phi_1$ and $w\lambda$ with coefficients $\mu S(\lambda_\mu^{+0}, \bar{g})$ because $\zeta^2\Phi \in S(1, \bar{g})$. Therefore $\{\Psi, b\phi_1\} \in \mu S(\lambda_\mu^{+0}, \bar{g})O(E)$. Recall $\Gamma = r + 2\omega\rho^{-2}$ and note $\text{supp } r \subset \text{supp } \chi$. With $B_1 = \zeta_+^2 a^2\hat{\theta}\phi_1 r\{\phi_1, \hat{\phi}_2\}$ taking Lemma 5.1 into account we can prove that

$$\{\Psi, B_1\} = \zeta_+(a_1w^{-1/2}\hat{\theta}\phi_1 + a_2\phi_1 + a_3\mu w^{-2}\hat{\theta})$$

where $a_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$. Here it is obvious that the supports of a_j are contained in that of ζ_+ . By similar arguments we get with $B_2 = 2\zeta_+^2 a^2\hat{\theta}\phi_1\omega\rho^{-2}\{\phi_1, \hat{\phi}_2\}$

$$\{\Psi, B_2\} = \zeta_+(\tilde{a}_1w^{-1/2}\hat{\theta}\phi_1 + \tilde{a}_2\phi_1 + \tilde{a}_3\mu w^{-2}\hat{\theta})$$

where $\tilde{a}_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$. This proves the assertion. \square

Proposition 6.1 *We have*

$$\begin{aligned} q\#T &= T\#(q - in\{q, \Psi\} - n^2\{\Psi, \{q, \Psi\}\} + i(a_1\mu w^{-3/2}\hat{\theta} \\ &\quad + a_2w^{1/2}\hat{\theta}\lambda + a_3\hat{\theta}\phi_1) + \mu S(\lambda_\mu^{+0}, \bar{g})O(E) \end{aligned}$$

where $a_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$ are real valued and $\text{supp } a_j \subset \text{supp } \zeta_+$.

Proof: From Lemma 6.3 it is clear that $T^{-1}T_{(\beta)}^{(\alpha)}\{q, \Psi\}_{(\alpha)}^{(\beta)}$ is $c_1\hat{\theta}\phi_1 + c_2w^{1/2}\hat{\theta}\lambda$ modulo $\mu^{1/2}S(\lambda_\mu^{+0}, \bar{g})O(E)$ because $\lambda_\mu^{-1/4} \in S(w^{1/2}, \bar{g})$ for $|\alpha + \beta| = 2$. Therefore we get

$$\begin{aligned} T\#\{q, \Psi\} &= T\{q, \Psi\} + nT\{\Psi, \{q, \Psi\}\}/2i + c_1\hat{\theta}\phi_1 \\ &\quad + c_2w^{1/2}\hat{\theta}\lambda + \mu S(\lambda_\mu^{+0}, \bar{g})O(E) \end{aligned}$$

where $c_i \in \mu S(\lambda_\mu^{+0}, \bar{g})$ is real. It is clear that $\text{supp } c_j \subset \text{supp } \zeta_+$. Thus we have $T\{q, \Psi\} = T\#(\{q, \Psi\} + n\{\Psi, \{\Psi, q\}\}/2i - c_1\hat{\theta}\phi_1 - c_2\mu w^{1/2}\hat{\theta}\lambda) + \mu S(\lambda_\mu^{+0}, \bar{g})O(E)$. From Lemma 6.4 it can be seen that $\Psi_{(\beta)}^{(\alpha)}\{\Psi, \{q, \Psi\}\}_{(\alpha)}^{(\beta)}$ for $|\alpha + \beta| = 1$ are written as $a_1\mu w^{-3/2}\hat{\theta} + a_2w^{1/2}\hat{\theta}\lambda + c_2\hat{\theta}\phi_1$ modulo $\mu S(\lambda_\mu^{+0}, \bar{g})O(E)$. This proves

$$T\{q, \Psi\} = T\#(\{q, \Psi\} + n\{\Psi, \{\Psi, q\}\}/2i + a_1\mu w^{-3/2}\hat{\theta} + a_2w^{1/2}\hat{\theta}\lambda + a_3\hat{\theta}\phi_1) + \mu S(\lambda_\mu^{+0}, \bar{g})O(E)$$

where $a_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$ with $\text{supp } a_j \subset \text{supp } \zeta_+$. This proves the assertion. \square

Taking $\hat{\theta}\Delta \in S(\lambda_\mu^{+0}, \bar{g})$ into account we have

Corollary 6.5 *We have*

$$\begin{aligned} \text{Im } \tilde{Q} = T_2 - \nu n \zeta_+^2 a^2 \hat{\theta} \phi_1 \Gamma\{\phi_1, \hat{\phi}_2\} + a_1 \mu w^{-3/2} \hat{\theta} + a_2 w^{1/2} \hat{\theta} \lambda \\ + a_3 \hat{\theta} \phi_1 + c_1 O(E) + c_2 O(E) + \mu S(\lambda_\mu^{+0}, \bar{g}) O(E) \end{aligned}$$

where $a_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$ are real valued of which support is contained in $\text{supp } \zeta_+$ and $c_1 = \zeta \chi c'_1$ with $c'_1 \in \mu S(w^{-1/2} \lambda_\mu^{+0}, \bar{g})$ and $c_2 = \zeta c'_2$ with $c_2 \in \mu S(\rho^{-1/2}, \bar{g})$.

Corollary 6.6 *We have*

$$\text{Re } \tilde{Q} = q + T_1 + \zeta_+(a_1 \mu w^{-2} \hat{\theta} + a_2 w^{-1/2} \hat{\theta} \phi_1 + a_3 \phi_1) + \mu S(\lambda_\mu^{+0}, \bar{g}) O(E)$$

where $a_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$.

7 Estimate $((\text{Re } \tilde{Q} - T_1 + \bar{\kappa} \mu \lambda)u, u)$

Here we write $q + T_1 = q + \bar{\kappa} \mu \lambda + (T_1 - \bar{\kappa} \mu \lambda)$ and instead of q we consider $q + \bar{\kappa} \mu \lambda$ with $\bar{\kappa} > 0$ in (3.10). In this section we study $((\text{Re } \tilde{Q} - T_1 + \bar{\kappa} \mu \lambda)u, u)$. Without restrictions we can assume $\bar{\kappa} = 1$.

Proposition 7.1 *Let $c_\pm \in S(1, \bar{g})$ be real. Then we have*

$$\begin{aligned} (7.1) \quad C((q + \mu \lambda)u, u) \geq \sum (\|c_\pm \zeta_\pm |\hat{\theta}|^{1/2} \phi_1 u\|^2 + |(c_\pm \zeta_\pm^2 |\hat{\theta}| \phi_1^2 u, u)|) \\ + |(\phi_2^2 u, u)| + |(w \phi_1^2 u, u)| + |(\omega \phi_1^2 u, u)| \\ + |(w^2 \lambda^2 u, u)| + |(\omega^2 \lambda^2 u, u)| + \|O(E)u\|^2. \end{aligned}$$

Proof: One can write

$$Ma^2 \zeta \hat{\theta} \phi_1^2 - (c_+ \zeta_+^2 \hat{\theta} + c_- \zeta_-^2 |\hat{\theta}|) \phi_1^2 = H_+^2 \zeta_+^2 \hat{\theta} \phi_1^2 + H_-^2 \zeta_-^2 |\hat{\theta}| \phi_1^2$$

with $H_+ = (Ma^2 \nu - c_+)^{1/2}$ and $H_- = (Ma^2 \hat{h} - c_-)^{1/2}$ where $M > 0$ is chosen so that $Ma^2 \nu - c_+ \geq c$, $Ma^2 \hat{h} - c_- \geq c > 0$. Since $\zeta_\pm |\hat{\theta}|^{1/2} \in S(|\hat{\theta}|^{1/2}, \bar{g})$ by

Lemma 3.1 noting $H_{\pm} \in S(1, \bar{g})$ we can write

$$\begin{aligned} & \zeta_{\pm} |\hat{\theta}|^{1/2} \phi_1 H_{\pm} \# \zeta_{\pm} |\hat{\theta}|^{1/2} \phi_1 H_{\pm} - \zeta_{\pm}^2 |\hat{\theta}| \phi_1^2 H_{\pm}^2 \\ &= \sum_{|\alpha+\beta|=2} C_{\alpha\beta} (\zeta_{\pm} |\hat{\theta}|^{1/2} \phi_1 H_{\pm})_{(\beta)}^{(\alpha)} (\zeta_{\pm} |\hat{\theta}|^{1/2} \phi_1 H_{\pm})_{(\alpha)}^{(\beta)} \\ &= b_1 w^{-3} \hat{\phi}_1^2 + b_2 w^{-5/2} \hat{\phi}_1 \end{aligned}$$

modulo $\mu^2 S(w^{-2}, \bar{g})$ where $b_i \in \mu^2 S(1, \bar{g})$. Write $b_1 w^{-3} \hat{\phi}_1^2 = c_1 w^{1/2} \phi_1 \# w^{1/2} \phi_1 + R_1$ and $b_2 w^{-5/2} \hat{\phi}_1 = \mu c_2 w^{-1} \# w^{1/2} \phi_1 + R_2$ with $c_i \in S(1, \bar{g})$ and $R_i \in \mu S(w^{-2}, \bar{g})$ we conclude

$$(7.2) \quad \sum |(c_{\pm} \zeta_{\pm}^2 |\hat{\theta}| \phi_1^2 u, u)| \leq M(a^2 \tilde{\zeta} \hat{\theta} \phi_1^2 u, u) + C(\|w^{1/2} \phi_1 u\|^2 + \mu^2 \|w^{-1} u\|^2).$$

Similarly $c_{\pm}^2 \zeta_{\pm}^2 |\hat{\theta}| \phi_1^2$ can be written

$$c_{\pm} \zeta_{\pm} |\hat{\theta}|^{1/2} \phi_1 \# c_{\pm} \zeta_{\pm} |\hat{\theta}|^{1/2} \phi_1 + b_{\pm} \mu^2 w^{-5/2} \hat{\phi}_1 + b'_{\pm} \mu^2 w^{-3} \hat{\phi}_1^2 + R$$

with $R \in S(w^{-2}, \bar{g})$. Thus $\|c_{\pm} \zeta_{\pm} |\hat{\theta}|^{1/2} \phi_1 u\|^2$ is estimated also by the right-hand side of (7.2).

We next study $\tilde{q} = \phi_2^2 + \chi_2 a^2 \phi_1^4 \lambda^{-2} + \mu \lambda$. If $\chi_2 \neq 1$ so that $\hat{\phi}_1^2 \leq d_3 w$ it is clear $\hat{\phi}_1^4 \leq C(\hat{\phi}_2^2 + \lambda_{\mu}^{-1})$ and then noting $\lambda \lambda_{\mu}^{-1} = \mu$ we have $\tilde{q} \geq c \hat{\phi}_1^4 \lambda^2$ with some $c > 0$. If $\chi_2 = 1$ this inequality is obvious. Since $\phi_1^4 \lambda^{-2} + \mu \lambda = \lambda^2 \omega^2$ and $\phi_2^2 + \mu \lambda = w^2 \lambda^2$ it is obvious $\tilde{q} \geq c(w^2 + \omega^2) \lambda^2$ with some $c > 0$. Let us set $\tilde{q} - c \omega^2 \lambda^2 = F^2$ with $F = \lambda(\tilde{q} \lambda^{-2} - c \omega^2)^{1/2} \in S(\lambda, \bar{g})$. If we note $\chi_2 a^2 \hat{\phi}_1^4 \in S(w^2, g)$ and $\omega \in S(\omega, G_1)$ with $G_1 = \omega^{-1/2} |dx|^2 + \omega^{-1/2} \langle \xi' \rangle_{\mu}^{-2} |d\xi'|^2$ then it is not difficult to see that $F^2 = F \# F + R$ with $R \in \mu^2 S(w^{-1} + \omega^{-1}, \bar{g})$. Thus we conclude that

$$(\tilde{q} u, u) \geq c(\omega^2 \lambda^2 u, u) - C \mu^4 (\|\omega^{-1} u\|^2 + \|w^{-1} u\|^2) - C \|u\|^2.$$

Repeating a similar argument we get $(\tilde{q} u, u) \geq c \|\hat{\phi}_1^2 \lambda u\|^2 - C \mu^4 \|w^{-1} u\|^2 - C \|u\|^2$. Since $\omega^2 \lambda^2 = \omega \lambda \# \omega \lambda + R$ with $R \in \mu^2 S(\omega^{-1}, \bar{g})$ and hence $(\omega^2 \lambda^2 u, u) \geq \|\omega \lambda u\|^2 - C(\mu^4 \|\omega^{-1} u\|^2 + \|u\|^2)$. Recalling $\phi_2^2 + \mu \lambda = w^2 \lambda^2$ similar arguments show

$$((\phi_2^2 + \mu \lambda) u, u) \geq c(w^2 \lambda^2 u, u) + \|w \lambda u\|^2 - C(\mu^4 \|w^{-1} u\|^2 + \|u\|^2).$$

Noting $\mu \lambda \leq w^2 \lambda^2 \in S(\lambda^2, g_0)$ we see $(w^2 \lambda^2 u, u) \geq \mu \|\lambda^{1/2} u\|^2 - C \|u\|^2$. On the other hand since one can write $w^{-1} = (w^{-1} \lambda^{-1/2}) \# \lambda^{1/2} + R$ with $R \in S(1, \bar{g})$ remarking $w^{-1} \lambda^{-1/2} \in \mu^{-1/2} S(1, \bar{g})$ we have $\|w^{-1} u\|^2 \leq C \mu^{-1} \|\lambda^{1/2} u\|^2 + C \|u\|^2$. Similarly we have $\|\omega^{-1} u\|^2 \leq C \mu^{-1} \|\lambda^{1/2} u\|^2 + C \|u\|^2$. Thus we get

$$(7.3) \quad \begin{aligned} & \mu^2 (\|w^{-1} u\|^2 + \|\omega^{-1} u\|^2) + \mu \|\lambda^{1/2} u\|^2 + \|w \lambda u\|^2 + \|\omega \lambda u\|^2 \\ & + |(w^2 \lambda^2 u, u)| + |(\omega^2 \lambda u, u)| + \|\hat{\phi}_1^2 \lambda u\|^2 \leq C(\tilde{q} u, u) + C \|u\|^2. \end{aligned}$$

Note $w^{1/2}\phi_1\#w^{1/2}\phi_1 = w\phi_1^2 + R$ with $R \in \mu^2 S(w^{-2}, \bar{g})$ and $w\phi_1^2 = \text{Re}(\lambda\hat{\phi}_1^2\#w\lambda) + R$ with $R \in \mu^2 S(w^{-2}, \bar{g})$ we have

$$\|w^{1/2}\phi_1 u\|^2 + |(w\phi_1^2 u, u)| \leq C\|O(E)u\|^2.$$

We get $\|\omega^{1/2}\phi_1 u\|^2 + |(\omega\phi_1^2 u, u)| \leq C(\|\lambda\hat{\phi}_1^2 u\|^2 + \|\omega\lambda u\|^2 + \mu^2\|\omega^{-1}u\|^2)$ by a repetition of similar arguments. It is easy to see $\|\phi_2 u\|^2 + |(\phi_2^2 u, u)| \leq C((\phi_2^2 + \mu\lambda)u, u) + \|u\|^2$ then we conclude the assertion by (7.3). \square

Corollary 7.1 *We have $\|\hat{\theta}\phi_1 u\|^2 + |(\hat{\theta}\phi_1^2 u, u)| \leq C((q + \mu\lambda)u, u) + C\|u\|^2$.*

Proof: Take $\eta(s) \in C_0^\infty(\mathbb{R})$ so that $\zeta_- + \zeta_+ + \eta = 1$. Thanks to Proposition 7.1 it suffices to prove $|(\eta\hat{\theta}\phi_1^2 u, u)| \leq C((q + \mu\lambda)u, u) + C\|u\|^2$. Note that one can write $\eta\hat{\theta}\phi_1^2 = c w\phi_1^2$ then the assertion follows immediately. \square

Lemma 7.2 *Let $\chi_0 = \chi_0(\hat{\phi}_1^2 w^{-1})$ with $\chi_0(s) \in C_0^\infty(\mathbb{R})$ which is 1 near $s = 0$. Then we have*

$$((1 - \chi_0)\zeta_\pm^2 |\hat{\theta}| w\lambda^2 u, u) \leq C((q + \mu\lambda)u, u) + C\|u\|^2.$$

Proof: Note that $Ma^2\tilde{\zeta}\hat{\theta}\phi_1^2 - (1 - \chi_0)(\zeta_+^2\hat{\theta} + \zeta_-^2|\hat{\theta}|)w\lambda^2 = H_+^2\zeta_+^2\hat{\theta}\phi_1^2 + H_-^2\zeta_-^2|\hat{\theta}|\phi_1^2$ where $H_+ = (Ma^2\nu - (1 - \chi_0)w\hat{\phi}_1^{-2})^{1/2}$ and $H_- = (Ma^2\hat{h} - (1 - \chi_0)w\hat{\phi}_1^{-2})^{1/2}$ which are in $S(1, \bar{g})$ taking $M > 0$ large. The rest of the proof is just a repetition of the proof of Proposition 7.1. \square

It is easy to check

$$(7.4) \quad \begin{aligned} & |(\zeta_+(a_1\mu w^{-2}\hat{\theta} + a_2w^{-1/2}\hat{\theta}\phi_1 + a_3\phi_1)u, u)| \\ & \leq Cn(\mu^2 + \gamma^{-1/2})\|\zeta_+w^{-1}\hat{\theta}u\|^2 + Cn^{-1}\|O(E)u\|^2. \end{aligned}$$

From Propositions 7.1 and 5.2 and Corollary 7.1 together with (7.4) we obtain

Proposition 7.2 *There exist $\gamma_0 > 0$, $\mu_0 > 0$, $n_0 > 0$ such that we have*

$$C((\text{Re}\tilde{Q} - T_1 + \bar{\kappa}\mu\lambda)u, u) + C\|\tilde{\Lambda}u\|^2 \geq |(\hat{\theta}\phi_1^2 u, u)| + \|\hat{\theta}\phi_1 u\|^2 + \|O(E)u\|^2.$$

for $\gamma \geq \gamma_0$, $0 < \mu < \mu_0$ and $n \geq n_0$. We have also

$$C(\lambda_\mu^{2\varepsilon}(\text{Re}\tilde{Q} - T_1 + \bar{\kappa}\mu\lambda)u, u) + C\|\lambda_\mu^\varepsilon\tilde{\Lambda}u\|^2 \geq \|\lambda_\mu^\varepsilon\hat{\theta}\phi_1 u\|^2 + \|\lambda_\mu^\varepsilon O(E)u\|^2.$$

8 Estimate $\text{Re}((\text{Re}\tilde{Q} - T_1 + \bar{\kappa}\mu\lambda)u, (\text{Im}\tilde{\lambda})u)$

Recall Lemma 4.3 which gives $\text{Im}\tilde{\lambda} = n\tilde{e}_1\Gamma\zeta_+^2\hat{\theta} + R_1$ with $R_1 \in S(\lambda_\mu^{+0}, \bar{g})$. Denote $\tilde{q} = \phi_2^2 + \chi_2 a^2 \phi_1^4 \lambda^{-2} + \mu\lambda$ again. Note $\text{Re}(\tilde{e}_1\hat{\theta}\zeta_+^2\Gamma\#\tilde{q}) = \tilde{e}_1\zeta_+^2\hat{\theta}\tilde{q}\Gamma + R$ with $R \in \mu S(\lambda^{1+0}, \bar{g})$ since $\Gamma \in S(w^{-1}\lambda_\mu^{+0}, \tilde{g})$ and $\phi_2^2 + \chi_2 a^2 \phi_1^4 \lambda^{-2} \in S(w^2\lambda^2, \tilde{g})$. Thus noting $|(Ru, u)| \leq C\mu\|\lambda^{1/2+0}u\|^2$ we get

$$\text{Re}(\tilde{q}u, \tilde{e}_1\hat{\theta}\zeta_+^2\Gamma u) \geq (\tilde{e}_1\zeta_+^2\hat{\theta}\tilde{q}\Gamma u, u) - C\|\lambda_\mu^{+0}O(E)u\|^2.$$

Write $M\tilde{e}_1\zeta_+^2\hat{\theta}\tilde{q}\Gamma - \mu\zeta_+^2\hat{\theta}w^2\lambda^2\Gamma = H\#(M\tilde{e}_1\tilde{q}w^{-2}\lambda^{-2} - \mu)\Gamma\#H + R$ with $H = \zeta_+\hat{\theta}^{1/2}w\lambda$ and $R \in \mu S(\lambda^{1+0}, \bar{g})$. Since $0 \leq (M\tilde{e}_1\tilde{q}w^{-2}\lambda^{-2} - \mu)\Gamma \in \mu S(w^{-1}\lambda_\mu^{+0}, \bar{g}) \subset \mu S_{3/4, 1/2}^{1/2+0}$ then from the Fefferman-Phong inequality it follows that

$$M(\tilde{e}_1\zeta_+^2\hat{\theta}\tilde{q}\Gamma u, u) - \mu(\zeta_+^2\hat{\theta}w^2\lambda^2\Gamma u, u) \geq -C\|\lambda_\mu^{+0}O(E)u\|^2.$$

Since $\zeta_+^2\hat{\theta}w^2\lambda^2\Gamma = H\#\Gamma\#H + R$ with $R \in \mu S(\lambda^{1+0}, \bar{g})$ taking $(\zeta_+^2 - \zeta^2)\hat{\theta} \in S(w, g)$ into account we conclude

$$(8.1) \quad \begin{aligned} & \mu(\Gamma(\zeta\hat{\theta}^{1/2}w\lambda u), \zeta_+\hat{\theta}^{1/2}w\lambda u) + \mu|(\zeta^2\hat{\theta}w^2\lambda^2\Gamma u, u)| \\ & \leq M\operatorname{Re}(\tilde{q}u, \tilde{e}_1\hat{\theta}\zeta_+^2\Gamma u) + C\|\lambda_\mu^{+0}O(E)u\|^2. \end{aligned}$$

Noticing $\Gamma = r + 2\omega\rho^{-2}$ and $\omega^s r \in S(w^{s-1}\lambda_\mu^{+0}, \bar{g})$ for $s \geq 0$ a repetition of similar argument for ω instead of w shows (8.1) where w is replaced by ω . It is easy to check that $w + \omega^3\rho^{-2} \geq c\rho$ with some $c > 0$. Since $(w^2 + \omega^2)\Gamma \geq \chi^2w + \omega^3\rho^{-2}$ and $C\omega \geq \rho \geq \omega$ on the support of $1 - \chi^2$ we see easily that

$$C\rho \geq (w^2 + \omega^2)\Gamma \geq c\rho$$

with some $c > 0$. Then applying the Fefferman-Phong inequality one obtains $(\zeta^2\hat{\theta}(w^2 + \omega^2)\lambda^2\Gamma u, u) \geq c\|\zeta\hat{\theta}^{1/2}\rho^{1/2}\lambda u\|^2 - C\|\lambda_\mu^{+0}O(E)u\|^2$. Thus

Lemma 8.1 *We have*

$$\mu|(\zeta^2\hat{\theta}\rho\lambda^2u, u)| + \mu\|\zeta\hat{\theta}^{1/2}\rho^{1/2}\lambda u\|^2 \leq C\operatorname{Re}(\tilde{q}u, \tilde{e}_1\hat{\theta}\zeta_+^2\Gamma u) + C\|\lambda_\mu^{+0}O(E)u\|^2.$$

We turn to $\operatorname{Re}(a^2\tilde{\zeta}\hat{\theta}\phi_1^2u, \tilde{e}_1\zeta_+^2\hat{\theta}\Gamma u)$. Since $\Gamma = r + 2\omega\rho^{-2}$ and $r\hat{\theta}^2 \in S(w, \bar{g})$ and $\omega\rho^{-2}\hat{\phi}_1^2 \in S(1, \bar{g})$ we see that $\operatorname{Re}(\tilde{e}_1\zeta_+^2\hat{\theta}\Gamma\#a^2\tilde{\zeta}\hat{\theta}\phi_1^2)$ is

$$\nu\tilde{e}_1\zeta_+^4\hat{\theta}^2a^2\phi_1^2\Gamma + \sum_{|\alpha+\beta|=2} \frac{(-1)^{|\beta|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!} (\tilde{e}_1\zeta_+^2\hat{\theta}\Gamma)_{(\beta)}^{(\alpha)} (a^2\tilde{\zeta}\hat{\theta}\phi_1^2)_{(\alpha)}^{(\beta)} + R$$

with $R \in \mu S(\lambda, \bar{g})$. Consider $(\tilde{e}_1\zeta_+^2\hat{\theta})_{(\beta_2)}^{(\alpha_2)}\Gamma_{(\beta_1)}^{(\alpha_1)}(a^2\tilde{\zeta}\hat{\theta})_{(\alpha'')}^{(\beta'')}(\phi_1^2)_{(\alpha')}^{(\beta')}$ for $|\alpha + \beta| = 2$. By Lemma 3.3 it is not difficult to see that we can write such a term as

$$(8.2) \quad \begin{aligned} & \nu c\zeta_+^2w\lambda^2\hat{\theta}^2 + \zeta_+(c_{21}w^{-1/2}\lambda\hat{\theta} + c_{22}w^{-1}\phi_1\hat{\theta} + c_{23}w^{-3/2}\hat{\theta}\phi_1^2\lambda) \\ & + (c_{31}w^{-1/2}\phi_1 + c_{32}w^{-1}\hat{\phi}_1^2\lambda) \end{aligned}$$

with $c \in \mu S(1, \bar{g})$ and $c_{ij} \in \mu^2 S(1, g)$. One can estimate the last term applying Proposition 7.1. The second term can be estimated thanks to Propositions 5.1 and 7.1. Indeed writing $c_{23}\zeta_+\hat{\theta}\phi_1^2\lambda = \operatorname{Re}(c_{23}\zeta_+w^{-3/2}\hat{\theta}\#\hat{\phi}_1^2\lambda) + R$ with $R \in S(w^{1/2}\lambda, \bar{g})$ we have

$$|\operatorname{Re}(c_{23}\zeta_+w^{-3/2}\hat{\theta}\phi_1^2\lambda u, u)| \leq C\mu^2\gamma^{-1/2}\|\zeta_+w^{-3/2}\hat{\theta}u\|^2 + C\gamma^{1/2}\|O(E)u\|^2.$$

To estimate the first term in (8.2) choosing $\nu > 0$ small we write $\zeta_+^2\hat{\theta}w\lambda^2 - \nu c\zeta_+^2w\hat{\theta}^2\lambda^2 = H\#H + R$ with $H = \zeta_+\hat{\theta}^{1/2}w^{1/2}\lambda(1 - \nu c\hat{\theta})^{1/2}$ and $R \in S(w^2\lambda^{2+0}, \bar{g})$ and apply Lemma 8.1. We now prove

Lemma 8.2 *There are $c > 0$ and $\nu_0 > 0$ such that we have*

$$(8.3) \quad \begin{aligned} & \operatorname{Re}(a^2 \tilde{\theta} \phi_1^2 u, \tilde{e}_1 \zeta_+^2 \hat{\theta} \Gamma u) \geq c \nu \mu (\Gamma(\zeta_+ \hat{\theta} \phi_1) u, \zeta_+ \hat{\theta} \phi_1 u) \\ & - C(\mu^3 n + \gamma^{-1/2}) \|\zeta w^{-3/2} \hat{\theta} u\|^2 - C(\mu n^{-1} + \gamma^{-1/2}) \|\zeta w^{1/2} \hat{\theta} \lambda u\|^2 \\ & - C \|\hat{\theta} \phi_1 u\|^2 - C \gamma^{1/2} (\|\zeta w^{-1} \hat{\theta} u\|^2 + \|O(E) u\|^2) \end{aligned}$$

for $0 < \nu \leq \nu_0$.

Proof: It remains to estimate $\nu \operatorname{Re}(\tilde{e}_1 \zeta_+^4 \hat{\theta}^2 a^2 \phi_1^2 \Gamma u, u)$ from below. Since $(\zeta_+^4 - \zeta_+^2) \hat{\theta}^2 \phi_1^2 \Gamma \in S(w \phi_1^2 \lambda_\mu^{+0}, \bar{g})$ it suffices to study $\nu \operatorname{Re}(\tilde{e}_1 \zeta_+^2 \hat{\theta}^2 a^2 \phi_1^2 \Gamma u, u)$. Note that

$$\begin{aligned} \operatorname{Re}(\zeta_+ \hat{\theta} \phi_1 \# \tilde{e}_1 a^2 \Gamma \# \zeta_+ \hat{\theta} \phi_1) &= \tilde{e}_1 \zeta_+^2 \hat{\theta}^2 a^2 \phi_1^2 \Gamma - \sum \frac{(-1)^{|\beta_1 + \beta_2 + \beta_3|}}{4 \alpha_1! \beta_1! \dots \beta_3!} \\ &\quad \times (\zeta_+ \hat{\theta} \phi_1)_{(\beta_1 + \beta_2)}^{(\alpha_1 + \alpha_2)} (\tilde{e}_1 a^2 \Gamma)_{(\alpha_1 + \beta_3)}^{(\beta_1 + \alpha_3)} (\zeta_+ \hat{\theta} \phi_1)_{(\alpha_2 + \alpha_3)}^{(\beta_2 + \beta_3)} + R \end{aligned}$$

where the sum is taken over $|\alpha_1 + \beta_1 + \dots + \beta_3| = 2$ and $R \in \mu^2 S(\lambda^{1+0}, \bar{g})$ which follows from Lemma 3.3. Here it can be checked that the second term is written as

$$c_1 \zeta^2 w^{-1} \lambda \hat{\theta}^2 + c_2 \zeta w^{-1} \hat{\theta} \phi_1 + c_3 w^{-1} \hat{\phi}_1^2 \lambda + c_4 w^{-1/2} \phi_1 + c_5 w^{-1/2} \zeta \hat{\theta} \lambda$$

with $c_i \in \mu^2 S(\lambda_\mu^{+0}, \bar{g})$ modulo $\mu^2 S(w^{-1} \lambda_\mu^{+0}, \bar{g})$. To estimate the first term let us write $c_1 \zeta^2 w^{-1} \lambda \hat{\theta}^2 = \operatorname{Re}(c_1 \zeta w^{-3/2} \hat{\theta} \# \zeta w^{1/2} \lambda \hat{\theta}) + R$ with $R \in \mu^2 S(\lambda^{1+0}, \bar{g})$. Then one can estimate $|\operatorname{Re}(c_1 \zeta^2 w^{-1} \hat{\theta}^2 u, u)|$ by

$$C \mu^3 n \|\zeta w^{-3/2} \hat{\theta} u\|^2 + C \mu n^{-1} \|\zeta w^{1/2} \lambda \hat{\theta} u\|^2 + C \|\lambda_\mu^{+0} O(E) u\|^2.$$

It is easy to see that $|((c_2 \zeta w^{-1} \hat{\theta} \phi_1 + c_3 w^{-1} \hat{\phi}_1^2 \lambda + c_4 w^{-1/2} \phi_1 + c_5 w^{-1/2} \zeta \hat{\theta} \lambda) u, u)|$ is bounded by $C \gamma^{-1/2} (\|\zeta w^{-3/2} \hat{\theta} u\|^2 + \|\zeta w^{1/2} \hat{\theta} \lambda u\|^2) + C \gamma^{1/2} \|O(E) u\|^2$. To end the proof it suffices to apply the Fefferman-Phong inequality to obtain

$$\operatorname{Re}(\tilde{e}_1 a^2 \Gamma(\zeta_+ \hat{\theta} \phi_1 u), \zeta_+ \hat{\theta} \phi_1 u) \geq c \mu \operatorname{Re}(\Gamma(\zeta_+ \hat{\theta} \phi_1 u), \zeta_+ \hat{\theta} \phi_1 u) - C \|\hat{\theta} \phi_1 u\|^2$$

because $\tilde{e}_1 a^2 - c \mu \geq 0$ with some $c > 0$. \square

Similar arguments proving Lemma 8.2 shows the estimate

$$\begin{aligned} \operatorname{Re}(a^2 \chi_2 \hat{\phi}_1^4 \lambda^2 u, \tilde{e}_1 \zeta_+^2 \hat{\theta} \Gamma u) &\geq -C \gamma^{-1/2} (\|\zeta_+ w^{-3/2} \hat{\theta} u\|^2 + \|\zeta_+ w^{1/2} \hat{\theta} \lambda u\|^2) \\ &\quad - C \gamma^{1/2} (\|\zeta_+ w^{-1} \hat{\theta} u\|^2 + \|O(E) u\|^2). \end{aligned}$$

We turn to consider

$$(8.4) \quad ((a_1 \mu w^{-2} \hat{\theta} + a_2 w^{-1/2} \hat{\theta} \phi_1 + a_3 \phi_1) u, (\operatorname{Im} \tilde{\lambda}) u).$$

To handle (8.4) we prepare a lemma.

Lemma 8.3 *We have*

$$\operatorname{Re}(\Gamma u, v) \leq (\Gamma v, v) + (\Gamma w, w) + C(\|\lambda_\mu^{+0} v\|^2 + \|\lambda_\mu^{+0} w\|^2).$$

Proof: Since $0 \leq \Gamma \in S(\lambda_\mu^{1/2+0}, \bar{g})$ it follows from the Fefferman-Phong inequality that $(\Gamma u, u) \geq -C\|\lambda_\mu^{+0}u\|^2$ with some $C > 0$. Thus with $L = \Gamma + C\lambda_\mu^{+0}$ we have $(Lu, u) \geq 0$ so that $|\operatorname{Re}(Lu, v)| \leq (Lu, u) + (Lv, v)$ which proves the assertion. \square

Write $\operatorname{Re}(\tilde{e}_1 \Gamma \zeta_+^2 \hat{\theta} \Gamma \# a_1 \mu w^{-2} \hat{\theta}) = \mu \operatorname{Re}(\Gamma \zeta_+ \hat{\theta} w \lambda \# a \zeta_+ w^{-1} \hat{\theta}) + R$ with $R \in S(w^{-2}, \bar{g})$ and apply Lemma 8.3 to get

$$\begin{aligned} |\operatorname{Re}(\tilde{e}_1 \Gamma \zeta_+^2 \hat{\theta} u, a_1 \mu w^{-2} \hat{\theta} u)| &\leq C \mu n^{-1} \operatorname{Re}(\Gamma \zeta_+ \hat{\theta} w \lambda u, \zeta_+ \hat{\theta} w \lambda u) \\ &\quad + C \mu n \operatorname{Re}(\Gamma a \zeta_+ w^{-1} \hat{\theta} u, a \zeta_+ w^{-1} \hat{\theta} u) \\ &\quad + C \mu (\|w \lambda u\|^2 + \mu^2 \|\zeta_+ w^{-1} \hat{\theta} u\|^2 + \|w^{-1} \lambda_\mu^{+0} u\|^2). \end{aligned}$$

Since $|\operatorname{Re}(\Gamma a \zeta_+ w^{-1} \hat{\theta} u, a \zeta_+ w^{-1} \hat{\theta} u)| \leq C \mu^2 (\|\zeta_+ w^{-3/2} \hat{\theta} u\|^2 + \|w^{-1} \lambda_\mu^{+0} u\|^2)$ we conclude

$$\begin{aligned} |\operatorname{Re}(\tilde{e}_1 \Gamma \zeta_+^2 \hat{\theta} u, a_1 \mu w^{-2} \hat{\theta} u)| &\leq C \mu n^{-1} \operatorname{Re}(\Gamma \zeta_+ \hat{\theta} w \lambda u, \zeta_+ \hat{\theta} w \lambda u) \leq C \|w \lambda u\|^2 \\ &\quad + C \mu^3 n \|\zeta_+ w^{-3/2} \hat{\theta} u\|^2 + C (\|\zeta_+ w^{-1} \hat{\theta} u\|^2 + \|w^{-1} \lambda_\mu^{+0} u\|^2). \end{aligned}$$

Similar arguments shows

$$\begin{aligned} |\operatorname{Re}(\tilde{e}_1 \Gamma \zeta_+^2 \hat{\theta} u, a_3 \phi_1 u)| &\leq \gamma^{-1/2} \operatorname{Re}(\Gamma \zeta_+ \hat{\theta} w \lambda u, \zeta_+ \hat{\theta} w \lambda u) \\ &\leq C \|w \lambda u\|^2 + C \gamma^{1/2} \mu^{-2} (\|w^{1/2} \phi_1 u\|^2 + \|w^{-1} \lambda_\mu^{+0} u\|^2). \end{aligned}$$

Repeating similar arguments we conclude that (8.4) is bounded by

$$\begin{aligned} &C(\mu n^{-1} + \gamma^{-1/2}) \operatorname{Re}(\Gamma \zeta_+ \hat{\theta} w \lambda u, \zeta_+ \hat{\theta} w \lambda u) + C \mu^2 n \|\zeta_+ w^{-3/2} \hat{\theta} u\|^2 \\ &\quad + C \gamma^{1/2} (\|\zeta_+ w^{-1} \hat{\theta} u\|^2 + \|O(E)u\|^2). \end{aligned}$$

We finally consider the term (qu, bu) with $b \in S(\lambda_\mu^{+0}, \bar{g})$. Noticing $\tilde{\zeta}' \hat{\theta}^{1/2} \in S(w^{1/2}, \bar{g})$ one sees

$$\operatorname{Re}(b \# a^2 \tilde{\zeta}^2 \hat{\theta} \phi_1^2) = \operatorname{Re}(ba \tilde{\zeta} \hat{\theta}^{1/2} \phi_1 \# a \tilde{\zeta} \hat{\theta}^{1/2} \phi_1) + O(E) \cdot O(E) + O(E)$$

and hence one obtains $|(qu, bu)| \leq C \|O(E)u\|^2$. We summarize

Proposition 8.1 *There is $c > 0$ and one can find $\gamma_0 > 0$, $\mu_0 > 0$, $n_0 > 0$, $\nu_0 > 0$ such that we have*

$$\begin{aligned} C\{ &\gamma((q + \mu \lambda)u, u) + \gamma^3 \|u\|^2 + \gamma \|\tilde{\Lambda} u\|^2 + \operatorname{Re}((\operatorname{Re} \tilde{Q} - T_1 + \bar{\kappa} \mu \lambda)u, \operatorname{Im} \tilde{\Lambda} u) \\ &+ \mu n \|\chi \zeta w^{-1/2} \tilde{\Lambda} u\|^2\} \geq cn \nu \mu (\Gamma(\zeta_+ \hat{\theta} \phi_1)u, \zeta_+ \hat{\theta} \phi_1 u) \\ &+ cn \mu |(\zeta^2 \hat{\theta} \rho \lambda^2 u, u)| + cn \mu \|\zeta \hat{\theta}^{1/2} \rho^{1/2} \lambda u\|^2 \end{aligned}$$

for $\gamma \geq \gamma_0$, $0 < \mu < \mu_0$, $n \geq n_0$ and $0 < \nu \leq \nu_0$.

9 Estimates of error terms

In this section we disregard error terms which are bounded by $\gamma^2 \|\lambda_\mu^{+0} u\|^2$ because we have $\gamma^3 \|\lambda_\mu^{3\varepsilon} u\|^2$ in (3.12). We estimate $\text{Re}(\tilde{\Lambda}u, (\text{Im } \tilde{Q} - T_2)u)$. Recall

$$\begin{aligned} \text{Im } \tilde{Q} - T_2 &= -\nu n \zeta_+^2 a^2 \hat{\theta} \phi_1 \Gamma\{\phi_1, \hat{\phi}_2\} + a_1 \mu w^{-3/2} \hat{\theta} + a_2 w^{1/2} \hat{\theta} \lambda \\ &\quad + a_3 \hat{\theta} \phi_1 + c_1 O(E) + c_2 O(E) + \mu S(\lambda_\mu^{+0}, \bar{g}) O(E). \end{aligned}$$

Thanks to Lemma 3.3 one can write

$$a^2 \zeta_+^2 \hat{\theta} \phi_1 \{\phi_1, \hat{\phi}_2\} \Gamma = \mu \zeta_+ \# \hat{a} \Gamma \# \zeta_+ \hat{\theta} \phi_1 + c_1 \zeta_+ \hat{\theta} \lambda + c_2 \zeta_+ \phi_1$$

with $\hat{a} = \mu^{-1} a^2 \{\phi_1, \hat{\phi}_2\}$ modulo $\lambda_\mu^{+0} O(E)$ where $c_i \in \mu S(\lambda_\mu^{+0}, \bar{g})$. Noticing $\zeta \zeta_+ = \zeta_+$ from Lemma 8.3 it follows that

$$\begin{aligned} \nu n \text{Re}(a^2 \zeta_+^2 \hat{\theta} \phi_1 \{\phi_1, \hat{\phi}_2\} \Gamma u, \tilde{\Lambda}u) &\leq \epsilon^{-1} n \nu^2 \mu (\hat{a} \Gamma(\zeta_+ \hat{\theta} \phi_1)u, (\zeta_+ \hat{\theta} \phi_1)u) \\ &\quad + \epsilon n \mu (\hat{a} \Gamma \zeta(\tilde{\Lambda}u), \zeta(\tilde{\Lambda}u)) + c n \nu \mu \|\zeta_+ \rho^{-1/2} \tilde{\Lambda}u\|^2 + c n \nu \mu \|\zeta_+ \rho^{1/2} \hat{\theta} \lambda u\|^2 \\ &\quad + C(\|\lambda_\mu^{+0} \hat{\theta} \phi_1 u\|^2 + \|\lambda_\mu^{+0} \tilde{\Lambda}u\|^2 + \|\lambda_\mu^{+0} O(E)u\|^2) \end{aligned}$$

where $\epsilon > 0$ will be determined later. We turn to estimate

$$((a_1 \mu w^{-3/2} \hat{\theta} + a_2 w^{1/2} \hat{\theta} \lambda + a_3 \hat{\theta} \phi_1)u, \tilde{\Lambda}u).$$

It is easy to see that this is bounded by

$$\begin{aligned} &C \gamma^{-1/2} (\|\zeta w^{-3/2} \hat{\theta} u\|^2 + \|\zeta w^{1/2} \hat{\theta} \lambda u\|^2) \\ &+ C \gamma^{1/2} (\|\tilde{\Lambda}u\|^2 + \|\hat{\theta} \phi_1 u\|^2 + \|\lambda_\mu^{+0} O(E)u\|^2). \end{aligned}$$

Finally we consider $|(c_1 O(E)u + c_2 O(E)u, \tilde{\Lambda}u)|$. Recalling Corollary 6.5 it is easily seen that this term is estimated by

$$C \gamma^{-1/2} (\|\zeta \chi w^{-1/2} \tilde{\Lambda}u\|^2 + \|\zeta \rho^{-1/2} \tilde{\Lambda}u\|^2) + C \gamma^{1/2} \|\lambda_\mu^{+0} O(E)u\|^2.$$

Noting $\|w^{1/2} \phi_1 u\|^2 + \|\omega^{1/2} \phi_1 u\|^2 \geq \|\rho^{1/2} \phi_1 u\|^2 - C \|O(E)u\|^2$ we summarize

Proposition 9.1 *The term $|\text{Re}(\tilde{\Lambda}u, (\text{Im } \tilde{Q} - T_2)u)|$ is bounded by*

$$\begin{aligned} &c \epsilon^{-1} n \nu^2 \mu (\Gamma(\zeta_+ \hat{\theta} \phi_1)u, (\zeta_+ \hat{\theta} \phi_1)u) + c \epsilon n \mu (\Gamma \zeta(\tilde{\Lambda}u), \zeta(\tilde{\Lambda}u)) \\ &\quad + (c n \nu \mu + C \gamma^{-1/2}) (\|\zeta \rho^{-1/2} \tilde{\Lambda}u\|^2 + \|\zeta \rho^{1/2} \hat{\theta} \lambda u\|^2) \\ &\quad + C \gamma^{-1/2} (\|\zeta w^{-3/2} \hat{\theta} u\|^2 + \|\zeta \chi w^{-1/2} \tilde{\Lambda}u\|^2) \\ &\quad + C \gamma^{1/2} (\|\hat{\theta} \phi_1 u\|^2 + \|\tilde{\Lambda}u\|^2 + \|\lambda_\mu^{+0} O(E)u\|^2) \end{aligned}$$

where $c > 0$ is independent of ϵ, ν, μ and γ .

We turn to consider the commutator $([D_0 - \operatorname{Re} \tilde{\lambda}, \operatorname{Re} \tilde{Q} - T_1]u, u)$. Recall

$$\xi_0 - \operatorname{Re} \tilde{\lambda} = \xi_0 - \phi_1 + \psi + n(b_2 \hat{\theta} + b_3 \hat{\phi}_1^2)w^{-1/2} + R_1$$

where $\psi = \tilde{\zeta} \hat{\theta} \phi_1 + \chi_2 \hat{\phi}_1^3 \lambda$ and $R \in S(\lambda_\mu^{+0}, \bar{g})$ and

$$\operatorname{Re} \tilde{Q} - T_1 = q + \zeta_+(a_1 \mu w^{-2} \hat{\theta} + a_2 w^{-1/2} \hat{\theta} \phi_1 + a_3 \phi_1) + \mu S(\lambda_\mu^{+0}, \bar{g}) O(E)$$

where $q = \phi_2^2 + 2\tilde{\zeta} a^2 \hat{\theta} \phi_1^2 + 2\chi_2 a^2 \hat{\phi}_1^4 \lambda^2$. Let us study $(\phi_2 \{\xi_0 - \phi_1 + \psi, \phi_2\} u, u)$. Taking (3.9) into account it suffices to estimate

$$\nu(c_1 \zeta_+^2 \hat{\theta} \phi_2 \lambda u, u), \quad (c_2 \phi_2 \hat{\phi}_1^2 \lambda u, u), \quad (c_3 \hat{\theta} \phi_2 \phi_1 u, u)$$

where $c_j \in \mu S(1, \bar{g})$. Write $\zeta_+^2 \hat{\theta} \phi_2 \lambda = (1 - \chi^2) \zeta_+^2 \hat{\theta} \phi_2 \lambda + \chi^2 \zeta_+^2 \hat{\theta} \phi_2 \lambda$ and consider $M \zeta_+^2 \hat{\theta} \phi_1^2 - (1 - \chi^2) \zeta_+^2 \hat{\theta} \phi_2 \lambda$ with a large positive constant M . Note

$$M^2 \zeta_+^2 \hat{\theta} \phi_1^2 - (1 - \chi^2) \zeta_+^2 \hat{\theta} \phi_2 \lambda = (M \phi_1)^2 F$$

where $0 \leq F = \zeta_+^2 \hat{\theta} (1 - (1 - \chi^2) \hat{\phi}_2 \hat{\phi}_1^{-2} / M^2) \in S(1, g)$. Writing $(M \phi_1)^2 F = \operatorname{Re}(M \phi_1 \# F \# M \phi_1) + R$ with $R \in S(w^{-2}, \bar{g})$ we obtain from the Fefferman-Phong inequality that

$$M^2 (\zeta_+^2 \hat{\theta} \phi_1^2 u, u) \geq ((1 - \chi^2) \zeta_+^2 \hat{\theta} \phi_2 \lambda u, u) - C \|O(E)u\|^2.$$

Consider now $2w \chi^2 \zeta_+^2 \hat{\theta} \lambda^2 - \chi^2 \zeta_+^2 \hat{\theta} \phi_2 \lambda = (w^{1/2} \lambda)^2 F$ with $0 \leq F = \chi^2 \zeta_+^2 \hat{\theta} (2 - \hat{\phi}_2 w^{-1}) \in S(1, \bar{g})$. Since $(w^{1/2} \lambda)^2 F = \operatorname{Re}(w^{1/2} \lambda \# F \# w^{1/2} \lambda) + R$ with $R \in \mu S(\lambda, \bar{g})$ from the Fefferman-Phong inequality one has

$$2(w \chi^2 \zeta_+^2 \hat{\theta} \lambda^2 u, u) \geq (\chi^2 \zeta_+^2 \hat{\theta} \phi_2 \lambda u, u) - C \|w^{1/2} \lambda^{1/2} u\|^2 - C \|O(E)u\|^2.$$

Here we note that $w^{1/2} \lambda^{1/2} \# w^{1/2} \lambda^{1/2} = w \lambda + R$ with $R \in S(1, \bar{g})$ and hence

$$2(w \chi^2 \zeta_+^2 \hat{\theta} \lambda^2 u, u) \geq (\chi^2 \zeta_+^2 \hat{\theta} \phi_2 \lambda u, u) - C \|O(E)u\|^2.$$

It is easy to see $|(c_3 \hat{\theta} \phi_1 \phi_2 u, u)| \leq C(\|\hat{\theta} \phi_1 u\|^2 + \|O(E)u\|^2)$ then we summarize

Lemma 9.1 *We have*

$$\begin{aligned} |(\{\xi_0 - \phi_1 + \psi, \phi_2^2\} u, u)| &\leq C \nu \mu (\chi^2 \zeta_+^2 \hat{\theta} w \lambda^2 u, u) \\ &\quad + C(\zeta_+^2 \hat{\theta} \phi_1^2 u, u) + C(\|\hat{\theta} \phi_1 u\|^2 + \|O(E)u\|^2). \end{aligned}$$

We next consider $\{\xi_0 - \phi_1 + \psi, \tilde{\zeta} a^2 \hat{\theta} \phi_1^2\}$ which is

$$\{\xi_0 - \phi_1, \tilde{\zeta} a^2 \hat{\theta} \phi_1^2\} + \{\tilde{\zeta} \hat{\theta} \phi_1, a^2 \phi_1\} \tilde{\zeta} \hat{\theta} \phi_1 + \{\chi_2 \hat{\phi}_1^3 \lambda, \tilde{\zeta} a^2 \hat{\theta} \phi_1^2\}.$$

It follows that $\{\chi_2 \hat{\phi}_1^3 \lambda, \tilde{\zeta} a^2 \hat{\theta} \phi_1^2\} = c_1 \hat{\phi}_1^4 \lambda^2$ and $\{\tilde{\zeta} \hat{\theta} \phi_1, a^2 \phi_1\} \tilde{\zeta} \hat{\theta} \phi_1 = c_2 \hat{\theta} \phi_1^2$ from Lemma 3.3. Since $\{\xi_0 - \phi_1, \tilde{\zeta} a^2 \hat{\theta} \phi_1^2\} = c_1 \hat{\theta} \phi_1^2 + c_2 \hat{\phi}_1^2 \lambda \phi_2 + c_3 \hat{\theta} \phi_1 \phi_2 + c_4 \hat{\phi}_1^4 \lambda^2$ by (3.1), (3.2) and Lemma 5.1 we get

$$\{\xi_0 - \phi_1 + \psi, \tilde{\zeta} a^2 \hat{\theta} \phi_1^2\} = c_1 \hat{\theta} \phi_1^2 + c_2 \hat{\phi}_1^2 \lambda \phi_2 + c_4 \hat{\theta} \phi_1 \phi_2 + c_5 \hat{\phi}_1^4 \lambda^2.$$

We then consider $\{\xi_0 - \phi_1 + \psi, \chi_2 a^2 \hat{\phi}_1^4 \lambda^2\}$ which is

$$\{\xi_0 - \phi_1, \chi_2 a^2 \hat{\phi}_1^4 \lambda^2\} + \{\tilde{\zeta} \hat{\theta} \phi_1, \chi_2 a^2 \hat{\phi}_1^4 \lambda^2\} + \{\chi_2 \hat{\phi}_1^3 \lambda, a^2 \hat{\phi}_1 \lambda\} \chi_2 \hat{\phi}_1^3 \lambda.$$

A repetition of similar arguments shows

$$\{\xi_0 - \phi_1 + \psi, \chi_2 a^2 \hat{\phi}_1^4 \lambda^2\} = c_1 \hat{\theta} \phi_1^2 + c_2 \hat{\phi}_1^4 \lambda^2 + \hat{\phi}_1^2 \lambda \phi_2.$$

Therefore $|(\{\xi_0 - \phi_1 + \psi, a^2 \tilde{\zeta} \hat{\theta} \phi_1^2 + a^2 \chi_2 \hat{\phi}_1^4 \lambda^2\} u, u)|$ is bounded by

$$C|(\hat{\theta} \phi_1^2 u, u)| + C\|O(E)u\|^2.$$

Denoting $\zeta_+ a_j$ by a_j we turn to check $\{\xi_0 - \phi_1 + \psi, a_2 w^{-1/2} \hat{\theta} \phi_1\}$ where $a_2 \in S(\lambda_\mu^{+0}, \bar{g})$ of which support is contained in $\text{supp } \zeta_+$. Remarking Lemmas 3.3 and 4.1 it is easy to see that

$$\{\xi_0 - \phi_1 + \psi, a_2 w^{-1/2} \hat{\theta} \phi_1\} = c_0 \zeta^2 w^{1/2} \lambda \hat{\theta} \phi_1 + c_1 \mu \zeta w^{-1/2} \hat{\theta} \lambda + c_2 \mu \lambda$$

with $c_j \in \mu S(\lambda_\mu^{+0}, \bar{g})$. Write $c_0 \zeta^2 w^{1/2} \lambda \hat{\theta} \phi_1 = \text{Re}(c_0 \zeta \hat{\theta}^{1/2} \phi_1 \# \zeta w^{1/2} \hat{\theta}^{1/2} \lambda) + R_1$ and $c_1 \zeta w^{-1/2} \hat{\theta} \lambda = \text{Re}(\zeta w^{-3/2} \hat{\theta} \# c_1 w \lambda) + R_2$ with $R_i \in \mu S(\lambda^{1+0}, \bar{g})$ we obtain the following estimate

$$\begin{aligned} |(c_0 \zeta^2 w^{1/2} \lambda \hat{\theta} \phi_1 u, u)| &\leq C \gamma^{-1/2} (\|\zeta w^{1/2} \hat{\theta}^{1/2} \lambda u\|^2 + \|\zeta w^{-3/2} \hat{\theta} u\|^2) \\ &\quad + C \gamma^{1/2} \|c_0 \zeta \hat{\theta}^{1/2} \phi_1 u\|^2 + C \|O(E)u\|^2. \end{aligned}$$

In order to estimate $\{\xi_0 - \phi_1 + \psi, a_1 w^{-2} \hat{\theta}\}$ we need to look at a_1 more carefully. Since $(w\phi)^{-1} \in S(\lambda_\mu, g)$ the main part of $\{F, \log \phi\}$ is $w^{-1} \{F, \hat{\phi}_2\}$ by (6.2). Therefore noticing (3.4) it is not difficult to see from the proof of Lemma 6.4 that a_1 has the form

$$(9.1) \quad f(\zeta_+)^{k_1} (\zeta'_+)^{k_2} (\chi)^{k_3} (\chi')^{k_4} \hat{\phi}_1^{\ell_1} \hat{\phi}_2^{\ell_2} w^{s_1} \omega^{s_2} \rho^{s_3} (\log \phi)^\epsilon$$

where $f \in S(1, g_0)$ and $k_i, \ell_i \in \mathbb{N}$ and $s_i \in \mathbb{R}$, $\epsilon = 0$ or 1 which verify

$$s_1 + s_2 + s_3 + \ell_1/2 + \ell_2 \geq 0$$

so that this is in $S(\lambda_\mu^{+0}, \bar{g})$. Here we examine that $\xi_0 - \phi_1 + \psi$ commutes better against such terms of the form (9.1) than against general symbol in $S(\lambda_\mu^{+0}, \bar{g})$.

Lemma 9.2 *Denote $\Lambda = \xi_0 - \phi_1 + \psi$ then $\{\Lambda, \hat{\phi}_1\}$, $\{\Lambda, \hat{\phi}_2\}$ and $\{\Lambda, \hat{\theta}\}$ is a linear combination of $\hat{\phi}_1$, $\hat{\phi}_2$ and $\hat{\theta}$ with $\mu S(1, \bar{g})$ coefficients. We denote this by $\{\Lambda, \hat{\phi}_1\} = \mu S(1, g_0) O(\Sigma)$ and so on.*

Proof: It follows easily from (3.1) and (3.2) that $\{\xi_0 - \phi_1, \hat{\phi}_1\}$, $\{\xi_0 - \phi_1, \hat{\phi}_2\}$ and $\{\xi_0 - \phi_1, \hat{\theta}\}$ are $O(\Sigma)$. Write $\psi = (\tilde{\zeta} \hat{\theta} + \chi_2 \hat{\phi}_1^2) \phi_1$ and note Lemma 3.3 then the desired assertion for $\{\psi, \hat{\phi}_1\}$, $\{\psi, \hat{\phi}_2\}$ and $\{\psi, \hat{\theta}\}$ follows immediately. \square

Corollary 9.3 *One can write $\{\Lambda, w^{-1}\} = \mu S(w^{-1}, \bar{g}) + \mu S(w^{-2}, \bar{g})O(\Sigma)$ and $\{\Lambda, \omega^{-1}\} = \mu S(\omega^{-1}, \bar{g}) + \mu S(\omega^{-3/2}, \bar{g})O(\Sigma)$ and that $\{\Lambda, \rho^{-1}\} = \mu S(w^{-1}, \bar{g}) + \mu S(w^{-2}, \bar{g})O(\Sigma) + S(\omega^{-3/2}, \bar{g})O(\Sigma)$. We have also $\{\Lambda, \zeta\} = c_1 w^{-1} \hat{\theta} + c_2 w^{-1/2}$ with $c_i \in \mu S(1, \bar{g})$ and the same holds for $\{\Lambda, \chi\}$.*

Let us consider $\{\Lambda, a_1\}$ where a_1 has the form (9.1) with $k_1 + k_2 \geq 1$. Since $(\chi)^{k_3}(\chi')^{k_4} \hat{\phi}_1 \in S(w^{1/2}, \bar{g})$ it follows from Lemma 9.2 and Corollary 9.3 that $\{\Lambda, a_1\}$ can be written as $c_0 w^{-1} \hat{\theta} + c_1 w^{-1/2} + c_2 \omega^{-1/2}$ with $c_i \in \mu S(\lambda_\mu^{+0}, \bar{g})$. Since $\omega^{-1/2} w^{-1/2} \in \mu^{-1/2} S(\lambda^{1/2}, \bar{g})$ then applying Lemma 9.2 and Corollary 9.3 again to $\{\Lambda, w^{-2} \hat{\theta}\}$ we conclude that

$$\mu\{\Lambda, a_1 w^{-2} \hat{\theta}\} = c_0 \mu^3 w^{-3} \hat{\theta}^2 + c_1 \mu^{5/2} w^{-3/2} \hat{\theta} \lambda^{1/2} + c_2 \mu^2 \phi_1 + O(E)$$

where $c_i \in S(\lambda_\mu^{+0}, \bar{g})$. Writing $c_0 w^{-3} \hat{\theta}^2 = \text{Re}(c_0 w^{-3/2} \hat{\theta} \# w^{-3/2} \hat{\theta}) + R$ with $R \in S(w^{-2}, \bar{g})$ and recalling that the support of c_i are contained in the support of ζ_+ we obtain the following estimate

$$(9.2) \quad |\mu(\{\xi_0 - \phi_1 + \psi, a_1 w^{-2} \hat{\theta}\} u, u)| \leq C(\mu^3 + \gamma^{-1/2}) \|\zeta w^{-3/2} \hat{\theta} u\|^2 + C\gamma^{1/2} \|O(E)u\|^2.$$

If we write $b_1 \omega^{-2} \hat{\theta} = (b_1 \omega^{-2} w^2) w^{-2} \hat{\theta}$ and $b_3 w^{-1} \omega^{-1} \hat{\theta} = (b_3 \omega^{-1} w) w^{-2} \hat{\theta}$ then $b_1 \omega^{-2} w^2$ and $b_3 \omega^{-1} w$ have the same form (9.1) and therefore we have the same estimate (9.2) for $|\mu(\{\xi_0 - \phi_1 + \psi, b_1 \omega^{-2} \hat{\theta} + b_3 w^{-1} \omega^{-1} \hat{\theta}\} u, u)|$. Since the estimate $|\mu(\{\xi_0 - \phi_1 + \psi, b_4 \phi_1\} u, u)| \leq C(\|\hat{\theta} \phi_1 u\|^2 + \|O(E)u\|^2)$ is easy we summarize

Proposition 9.2 *We have*

$$\begin{aligned} |([D_0 - \text{Re} \tilde{\lambda}, \text{Re} \tilde{Q}] u, u)| &\leq c\nu\mu(\chi^2 \zeta_+^2 \hat{\theta} w \lambda^2 u, u) \\ &+ (c\mu^3 + C\gamma^{-1/2}) \|\zeta w^{-3/2} \hat{\theta} u\|^2 + C(\zeta_+^2 \hat{\theta} \phi_1^2 u, u) \\ &+ C\gamma^{-1/2} (\|\zeta \rho^{1/2} \hat{\theta}^{1/2} \lambda u\|^2 + \|\zeta w^{-3/2} \hat{\theta} u\|^2) \\ &+ C\gamma^{1/2} (\|\hat{\theta} \phi_1 u\|^2 + \|\zeta \hat{\theta}^{1/2} \phi_1 u\|^2 + \|O(E)u\|^2) \end{aligned}$$

where $c > 0$ is independent of ν , μ and γ .

10 Lower order terms

We finally handle the lower order terms. By (3.11) one can write

$$T_2 = \mu c_0 \hat{\theta} \lambda + b_0 \hat{\theta} \phi_1 + b_1 \hat{\phi}_1^2 \lambda + b_2 \phi_2 + b_3 w^{1/2} \phi_1$$

with $b_j \in \mu S(1, g)$ where $c_0 = 0$ for $\hat{\theta} < 0$ by assumption. Write $c_0 \hat{\theta} \lambda = c_0 \zeta_+^2 \hat{\theta} \lambda + (1 - \zeta_+^2) c_0 \hat{\theta} \lambda$ where it is clear that we can write $(1 - \zeta_+^2) c_0 \hat{\theta} \lambda = b_4 w \lambda$. We examine that one can write

$$\begin{aligned} (1 - \chi^2) \zeta_+^2 \hat{\theta} \lambda &= \omega^{1/2} \rho^{-1} \zeta_+ \# \rho \omega^{-1/2} (1 - \chi^2) \zeta_+ \hat{\theta} \lambda \\ &+ \omega^{1/2} \rho^{-1} \zeta_+ \# c \rho \omega^{-1/2} \hat{\theta} \lambda + R \end{aligned}$$

with $c \in S(\lambda_\mu^{-1/4}, \bar{g})$ and $R \in S(\lambda^{1/2}, \bar{g})$. Moreover $\text{supp } c \subset \text{supp}(1 - \chi^2)$. Indeed since $\omega^{\pm 1/2} \rho^{\mp 1} \in S(\omega^{\pm 1/2} \rho^{\mp 1}, \bar{g})$ then $\omega^{1/2} \rho^{-1} \zeta_+ \# \rho \omega^{-1/2} (1 - \chi^2) \zeta_+ \hat{\theta} \lambda$ can be written as $c \zeta_+ \hat{\theta} \lambda + R$ with $c \in S(\lambda_\mu^{-1/4}, \bar{g})$ and $R \in \mu^{1/2} S(\lambda^{1/2}, \bar{g})$. Write $c \zeta_+ \hat{\theta} \lambda = \omega^{1/2} \rho^{-1} \zeta_+ \# c \rho \omega^{-1/2} \hat{\theta} \lambda + R$ again we get the desired assertion. This proves

$$\begin{aligned} |((1 - \chi^2) \zeta_+^2 \hat{\theta} \lambda u, \tilde{\Lambda} u)| &\leq C \gamma^{-1/2} \|\omega^{1/2} \rho^{-1} (\zeta_+ \tilde{\Lambda} u)\|^2 \\ &+ C \gamma^{1/2} (\|c \rho \omega^{-1/2} \hat{\theta} \lambda u\|^2 + \|\tilde{\Lambda} u\|^2 + \|O(E)u\|^2) \end{aligned}$$

with $c \in S(\lambda_\mu^{-1/4}, \bar{g})$ where $\text{supp } c \subset \text{supp}(1 - \chi^2)$. Now consider $\|c \rho \omega^{-1/2} \hat{\theta} \lambda u\|^2$. Note that $c \rho \omega^{-1/2} \in S(\omega^{1/2}, \bar{g})$ because if $c \neq 0$ then we have $C \hat{\phi}_1^2 \geq w$ and hence $C^2 \omega^2 \geq w^2 \geq \hat{\phi}_2^2$. Thus it is clear $\omega^2 \leq \hat{\phi}_2^2 + \omega^2 = \rho^2 \leq (C^2 + 1) \omega^2$ so that $\omega^{1/2} \leq \rho \omega^{-1/2} \leq (1 + C') \omega^{1/2}$. Hence it is easily seen that $c \rho \omega^{-1/2} \hat{\theta} \lambda \# c \rho \omega^{-1/2} \hat{\theta} \lambda = a \omega \lambda^{3/2} + R$ with $a \in S(1, \bar{g})$ and $R \in S(\lambda, \bar{g})$ so that $\|c \rho \omega^{-1/2} \hat{\theta} \lambda u\|^2 \leq C \|O(E)u\|^2$. We summarize

$$\begin{aligned} |((1 - \chi^2) \zeta_+^2 \hat{\theta} \lambda u, \tilde{\Lambda} u)| &\leq C \gamma^{-1/2} \|\rho^{-1} \omega^{1/2} \zeta_+ (\tilde{\Lambda} u)\|^2 \\ &+ C \gamma^{1/2} (\|\tilde{\Lambda} u\|^2 + \|O(E)u\|^2). \end{aligned}$$

We turn to study $(\chi^2 \zeta_+^2 \hat{\theta} \lambda u, \tilde{\Lambda} u)$. Let us write $\chi^2 \zeta_+^2 \hat{\theta} \lambda = \chi \zeta_+ w^{-1/2} \# \chi \zeta_+ w^{1/2} \hat{\theta} \lambda + c \hat{\theta} \lambda + R$ with $c \in S(\lambda_\mu^{-1/2}, \bar{g})$ and $R \in S(\lambda^{1/2}, \bar{g})$ and hence we have

$$\begin{aligned} |(\chi^2 \zeta_+^2 \hat{\theta} \lambda u, \tilde{\Lambda} u)| &\leq c n^{1/2} \|\chi \zeta_+ w^{-1/2} \tilde{\Lambda} u\|^2 \\ &+ c n^{-1/2} \|\zeta_+ \chi w^{1/2} \hat{\theta} \lambda u\|^2 + C (\|O(E)u\|^2 + \|\tilde{\Lambda} u\|^2). \end{aligned}$$

Since it is clear that $|((b_0 \hat{\theta} \phi_1 + b_1 \hat{\phi}_1^2 \lambda + b_2 \phi_2 + b_3 w^{1/2} \phi_1)u, \tilde{\Lambda} u)|$ is bounded by $C(\|\tilde{\Lambda} u\|^2 + \|\hat{\theta} \phi_1 u\|^2 + \|O(E)u\|^2)$ we get

Proposition 10.1 *We have*

$$\begin{aligned} |(T_2 u, \tilde{\Lambda} u)| &\leq (c \mu n^{1/2} + C \gamma^{-1/2}) \|\chi \zeta_+ w^{-1/2} \tilde{\Lambda} u\|^2 \\ &+ c \mu n^{-1/2} \|\chi \zeta_+ w^{1/2} \hat{\theta} \lambda u\|^2 + C \gamma^{-1/2} \|\rho^{-1} \omega^{1/2} \zeta \tilde{\Lambda} u\|^2 \\ &+ C \gamma^{1/2} (\|\tilde{\Lambda} u\|^2 + \|\lambda_\mu^{+0} O(E)u\|^2) \end{aligned}$$

with $c > 0$ independent of n, ν, μ and λ .

We turn to consider $((T_1 - \bar{\kappa} \mu \lambda)u, u)$. From Lemma 3.1 it follows that $\zeta_-^2 h |\hat{\theta}| \phi_1^2 = h^{1/2} \phi_1 \zeta_- |\hat{\theta}|^{1/2} \# h^{1/2} \phi_1 \zeta_- |\hat{\theta}|^{1/2} + R$ with $R \in \mu^2 S(w^{-2}, g)$. Noting $h^{1/2} \zeta_- |\hat{\theta}|^{1/2} \in S(w^{1/2}, g)$ we see

$$\phi_2 \# h^{1/2} \phi_1 \zeta_- |\hat{\theta}|^{1/2} - h^{1/2} \phi_1 \zeta_- |\hat{\theta}|^{1/2} \# \phi_2 = \{\phi_2, h^{1/2} \phi_1 \zeta_- |\hat{\theta}|^{1/2}\} / i + R$$

with $R \in \mu^2 S(w^{-3/2}, g)$. Here since $h = \mu \hat{c} \{\phi_1, \hat{\phi}_2\}^{-1}$ we have

$$\{h^{1/2} \phi_1 \zeta_- |\hat{\theta}|^{1/2}, \phi_2\} = \mu^{1/2} \zeta_- (\hat{c} \{\phi_1, \hat{\phi}_2\} |\hat{\theta}|)^{1/2} e + c \mu w^{1/2} \phi_1$$

with $c \in S(1, g)$ thanks to Lemma 3.1 because $\phi_2^{(\alpha)} \in \mu S(w, g)$ for $|\alpha| = 1$. Then the following estimate follows easily

$$\mu^{1/2}(\zeta_-(\hat{c}\{\phi_1, \hat{\phi}_2\}|\hat{\theta}|)^{1/2}e u, u) \leq (\phi_2^2 u, u) + (h\zeta_-^2|\hat{\theta}|\phi_1^2 u, u) + C\mu^{1/4}\|O(E)u\|^2$$

because $\|w^{-1/2}u\|^2 \leq C\mu^{-1/2}\|\lambda^{1/2}u\|^2$. From (3.10) it follows that

$$(10.1) \quad \mu^{1/2}\zeta_-(\hat{c}\{\phi_1, \hat{\phi}_2\}|\hat{\theta}|)^{1/2}e + T_1 \geq 2\bar{\kappa}\mu\lambda - C\mu w^{1/2}\lambda$$

with some $C > 0$. In fact if $\hat{\theta} \leq -b_3 w$ then $\zeta_- = 1$ and the assertion by (3.10). If $-b_3 w \leq \hat{\theta} \leq 0$ then we have $Cw^{1/2}\lambda \geq \mu^{-1/2}(\hat{c}\{\phi_1, \hat{\phi}_2\}|\hat{\theta}|)^{1/2}e$ and hence the assertion. Since $S(\lambda, G) \subset S_{1,1/2}^1$ the Fefferman-Phong inequality gives

$$\mu^{1/2}(\hat{c}\zeta_-(\{\phi_1, \hat{\phi}_2\}|\theta|)^{1/2}e u, u) + (T_1 u, u) \geq 2\bar{\kappa}\mu(\lambda u, u) - C\mu\|O(E)u\|^2.$$

We summarize

Proposition 10.2 *We have*

$$\begin{aligned} & (\phi_2^2 u, u) + (h\zeta_-^2|\hat{\theta}|\phi_1^2 u, u) + ((T_1 - \bar{\kappa}\mu\lambda)u, u) \\ & \geq \bar{\kappa}\mu(\lambda u, u) - C\mu^{1/4}\|O(E)u\|^2. \end{aligned}$$

Similarly $(\lambda_\mu^{2\varepsilon}\phi_2^2 u, u) + (\lambda_\mu^{2\varepsilon}h\zeta_-^2|\hat{\theta}|\phi_1^2 u, u) + (\lambda_\mu^{2\varepsilon}(T_1 - \bar{\kappa}\mu\lambda)u, u)$ is bounded from below by $\bar{\kappa}\mu(\lambda_\mu^{2\varepsilon}\lambda u, u) - C\mu^{1/4}\|\lambda_\mu^\varepsilon O(E)u\|^2$.

Finally we estimate $\text{Re}((T_1 - \bar{\kappa}\mu\lambda)u, (\text{Im}\tilde{\lambda})u)$. Since $\zeta_-\zeta_+ = 0$ then from (10.1) we see that $\text{Re}((T_1 - \bar{\kappa}\mu\lambda)u, (\text{Im}\tilde{\lambda})u)$ is bounded from below by

$$\text{Re}((\bar{\kappa}\mu\lambda - C\mu w^{1/2}\lambda)u, \tilde{e}_1\Gamma\zeta_+\hat{\theta}u) - C\|\lambda_\mu^{+0}O(E)u\|^2.$$

Note that $\text{Re}(\tilde{e}_1\Gamma\zeta_+\hat{\theta}\#(\bar{\kappa}\mu\lambda - C\mu w^{1/2}\lambda)) = \bar{\kappa}\mu\tilde{e}_1\Gamma\zeta_+\hat{\theta}\lambda + c\zeta_+w^{-1/2}\hat{\theta}\lambda + R$ with $c \in S(\lambda_\mu^{+0}, \bar{g})$ and $R \in \mu S(w^{-1}, \bar{g})$. Since $0 \leq \tilde{e}_1\Gamma\zeta_+\hat{\theta}\lambda \in S(w^{-1}\lambda^{1+0}, \bar{g})$ and noting $c\zeta_+w^{-1/2}\hat{\theta}\lambda = \text{Re}(\zeta_+w^{-3/2}\hat{\theta}\#cw\lambda) + R$ with $R \in S(\lambda, \bar{g})$ one can see that $\text{Re}(\tilde{e}_1\Gamma\zeta_+\hat{\theta}\#(\bar{\kappa}\mu\lambda - C\mu w^{1/2}\lambda))$ has a bound from below $-C(\gamma^{-1/2}\|\zeta_+w^{-3/2}\hat{\theta}u\|^2 + \gamma^{1/2}\|\lambda_\mu^{+0}O(E)u\|^2)$. Therefore we obtain

Lemma 10.1 *We have*

$$\text{Re}((T_1 - \bar{\kappa}\mu\lambda)u, (\text{Im}\tilde{\lambda})u) \geq -C\gamma^{-1/2}\|\zeta_+w^{-3/2}\hat{\theta}u\|^2 - C\gamma^{1/2}\|\lambda_\mu^{+0}O(E)u\|^2.$$

We first choose $\epsilon > 0$ small so that $c\epsilon n\mu(\Gamma(\zeta(\tilde{\Lambda}u), \zeta(\tilde{\Lambda}u)))$ in Proposition 9.1 can be controlled by the corresponding term in Proposition 4.1. We next choose $\nu > 0$ small so that $c\nu\mu\|\zeta\rho^{-1/2}\tilde{\Lambda}u\|^2$ and

$$c\epsilon^{-1}n\nu^2\mu(\Gamma(\zeta_+\hat{\theta}\phi_1)u, (\zeta_+\hat{\theta}\phi_1)u) + c\nu\mu\|\zeta\rho^{1/2}\hat{\theta}\lambda u\|^2$$

in Proposition 9.1 will be small against the corresponding terms in Propositions 4.1 and 8.1. We then choose n such that $\mu^3\|\zeta w^{-3/2}\hat{\theta}u\|^2$ in Proposition 9.2 can be controlled by Proposition 5.1 and $c\mu n^{1/2}\|\chi\zeta_+w^{-1/2}\tilde{\Lambda}u\|^2 +$

$c\mu n^{-1/2}\|\chi\zeta_+w^{1/2}\hat{\theta}\lambda u\|^2$ in Proposition 10.1 can be estimated by Propositions 4.1 and 8.1. Finally we choose $\mu > 0$ small enough and then $\gamma > 0$ enough large so that μn^4 to be small and $\gamma\mu^4$ to be large. Then combining Propositions 4.1, 5.1, 5.2, 7.2, 8.1, 9.1, 9.2, 10.1 and 10.2 we obtain a desired weighted energy estimates. Once we obtain energy estimates in order to conclude the well-posedness of the Cauchy problem it suffices to apply [17, Theorem 1.1].

References

- [1] A.Barbagallo and V.Esposito, A global existence and uniqueness result for a class of hyperbolic operators, *Ricerche Mat.* **63** (2014), 25-40.
- [2] R.Beals, Characterization of pseudodifferential operators and applications, *Duke Math. J.* **44** (1977), 45-57.
- [3] E.Bernardi and A.Bove, Geometric transition for a class of hyperbolic operators with double characteristics, *Japan J. Math.* **23** (1997), 1-87.
- [4] E.Bernardi, C.Parenti and A.Parmeggiani, The Cauchy problem for hyperbolic operators with double characteristics in presence of transition, *Comm. Partial Differential Equations* **37** (2012), 1315-1356.
- [5] V.Esposito, On the well posedness of the Cauchy problem for a class of hyperbolic operators with double characteristics, *Ricerche Mat.* **49** (2000), 221-239.
- [6] L.Hörmander, The Cauchy problem for differential equations with double characteristics, *J. Analyse Math.* **32** (1977), 118-196.
- [7] L.Hörmander, *The Analysis of Linear Partial Differential Operators, III*, Springer, Berlin-Heidergerg-New York-Tokyo, 1985.
- [8] V.Ja.Ivrii and V.M.Petkov, Necessary conditions for the Cauchy problem for non strictly hypebolic equations to be well posed, *Uspehi Mat. Nauk.* **29** (1974), 3-70.
- [9] V.Ja.Ivrii, Wave fronts of solutions of certain pseudo-differential equations, *Trans. Moscow Math. Soc.* **39** (1981), 49-86.
- [10] V.Ja.Ivrii, The well posedness of the Cauchy problem for non strictly hyperbolic operators III: The energy integral, *Trans. Moscow Math. Soc.* **34** (1978), 149-168.
- [11] G.Komatsu and T.Nishitani , Continuation of bicharacteristics for effectively hyperbolic operators, *Proc. Japan Acad.* **65**, Ser. A (1989), 109-112.
- [12] R.Melrose, The Cauchy problem and propagation of singularities, In: *Seminar on Nonliniar Partial Differential Equations* (ed. S.S.Chern), Math. Sci. Res. Inst. Publ., **2**, Springer, 1984, pp. 185-201.

- [13] T.Nishitani, Local energy integrals for effectively hyperbolic operators, I, J. Math. Kyoto Univ. **24** (1984), 625-658; II, 659-666.
- [14] T.Nishitani, Non effectively hyperbolic operators, Hamilton map and bicharacteristics, J. Math. Kyoto Univ. **44** (2004), 55-98.
- [15] T.Nishitani, On the Cauchy problem for noneffectively hyperbolic operators, a transition casse, In: Studies in Phase Space Analysis with Applications to PDEs (eds. M.Cicognani, F.Colombini and D.Del Santo), pp. 259–290, Birkhäuser, 2013.
- [16] T.Nishitani, On the Cauchy problem for hyperbolic operators with double characteristics, a transition casse, In: Fourier Analysis, Trends in Mathematics (eds. M.Ruzhansky and V.Turunen), pp. 311–334, Birkhäuser, 2014.
- [17] T.Nishitani, Local and microlocal Cauchy problem for non-effectively hyperbolic operators, J. Hyperbolic Differ. Equ. **11** (2014), 185-213.